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(NASA-CR-135425) CALCULATION OF THE FLOW
FIELD IN SUPERSONIC MIXED-COMPRESSION INLETS
AT ANGLE OF ATTACK USING THE
THREE-DIMENSIONAL METHOD OF CHARACTERISTICS
WITH DISCRETE SHOCK WAVE FITTING (Purdue

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**CALCULATION OF THE FLOW FIELD IN SUPERSONIC MIXED-
COMPRESSION INLETS AT ANGLE OF ATTACK USING THE
THREE-DIMENSIONAL METHOD OF CHARACTERISTICS
WITH DISCRETE SHOCK WAVE FITTING**

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16. Abstract The calculation of the flow field in supersonic mixed-compression aircraft inlets at angle of attack is accomplished using the method of characteristics for steady three-dimensional flow in conjunction with a discrete shock wave fitting procedure. The influence of molecular transport can be included in the computation by treating the viscous and thermal diffusion terms in the governing partial differential equations as correction terms in the method of characteristics scheme. The culmination of the present research is the development of a production type computer program which is capable of calculating the flow field in a variety of axisymmetric mixed-compression aircraft inlets. The results produced by the present analysis agree well with those produced by the two-dimensional method characteristics when axisymmetric flow fields are computed. For three-dimensional flow fields, the results of the present analysis agree well with experimental data except in regions of high viscous interaction and boundary layer removal. The present analysis does not compute the boundary layer or account for boundary layer bleed. Submitted as a thesis by Joseph Vadyak to Purdue University, West Lafayette, Indiana, in partial fulfillment of the requirement for the degree of Doctor of Philosophy, March 1978.			
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CALCULATION OF THE FLOW FIELD IN SUPERSONIC MIXED-COMPRESSION INLETS
AT ANGLE OF ATTACK USING THE THREE-DIMENSIONAL METHOD OF
CHARACTERISTICS WITH DISCRETE SHOCK WAVE FITTING*

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SUMMARY

An analysis has been developed for calculating the flow field in supersonic mixed-compression aircraft inlets at angle of attack using the three-dimensional method of characteristics with discrete shock wave fitting. This report describes the details of the analysis and presents some computational results.

The gas dynamic model is based on the assumptions of steady continuum flow, negligible body forces, a simple system in thermodynamic equilibrium, no mass diffusion, negligible radiative heat transfer, no internal heat generation other than viscous dissipation, and viscous and thermal diffusion effects of secondary importance. The viscous and thermal diffusion terms are treated as forcing functions, or correction terms, in the method of characteristics scheme. Pressure and density are specified as the primary thermodynamic properties, and the temperature, speed of sound, viscosity, and thermal conductivity are expressed in terms of pressure and density.

The system of governing equations is hyperbolic when the flow is supersonic. The equations for the characteristic surfaces and the compatibility equations applicable along these surfaces are derived. The characteristic surfaces are the stream surfaces, which are surfaces composed of streamlines, and the wave surfaces, which are surfaces tangent to a Mach conoid. The compatibility equations are expressed as directional derivatives along streamlines and bi-characteristics, which are the lines of tangency between a wave surface and a Mach conoid. The numerical integration procedure devised by D.S. Butler was employed to develop a numerical integration algorithm that is second-order accurate, explicit, and does not violate the domain of dependence of the differential equations.

The bow shock wave surrounding the forebody portion of the centerbody and the internal shock wave system inside of the inlet are determined by discrete

*Submitted to Purdue University, West Lafayette, Indiana, by the first author in partial fulfillment of the requirements for the degree of Doctor of Philosophy, March 1978.

shock wave fitting. The continuous flow field between shock waves is determined by the method of characteristics numerical integration procedure, and the flow properties across the shock waves are determined by the application of the Hugoniot jump conditions.

Unit processes were developed for interior field points, solid boundary points, field-shock wave points, and solid boundary-shock wave points. An inverse marching scheme is employed in which the solution is obtained on planes perpendicular to the axis of the centerbody and the cowl. The distance between successive solution planes is determined by the Courant-Friedrichs-Lewy stability criterion. Although the numerical integration procedure developed herein is capable of analyzing three-dimensional flows in three-dimensional geometries, only axisymmetric geometries at angle of attack were considered in the present investigation.

Selected computational results are presented for three categories of flow fields: external flow about the forebody, continuous internal flow, and internal flow in which the discrete internal shock wave system is computed. Both axisymmetric flow results and three-dimensional flow results are presented. For the internal flow field in which the shock waves have been fitted, some comparisons with experimental data are presented. Results of the present analysis are compared with those obtained by the two-dimensional method of characteristics for axisymmetric flows, and by a three-dimensional fixed grid finite difference shock capturing method.

The computational results support the following conclusions. The external flow field about a forebody can be accurately calculated if a bow shock wave of reasonable strength exists. For axisymmetric flows, the solution agrees well with results obtained by the two-dimensional method of characteristics. Except in regions of strong viscous interaction and boundary layer removal, the results of the present analysis agree well with experimental data. Good agreement is obtained between the present analysis and a finite difference shock capturing method. The present analysis, however, which discretely fits shock waves, provides better resolution of the shock waves.

SECTION I

INTRODUCTION

1. GENERAL

The purpose of this investigation was to develop a method for calculating the flow field in a supersonic mixed-compression aircraft inlet at nonzero angle of attack. A typical supersonic mixed-compression aircraft inlet is illustrated in Figure 1. Compression takes place both in the external flow about the forebody and in the internal flow inside the annulus. The free-stream velocity is supersonic, hence, a bow shock wave is generated at the forebody tip as shown. The internal shock wave emanates from the cowl lip. That shock wave makes a number of reflections with the centerbody and cowl before terminating in the divergence downstream of the geometric throat of the annulus. The flow is subsonic downstream of that location.

A major objective in the design of any aircraft inlet is to achieve maximum flow compression with a minimum reduction in stagnation pressure. Moreover, since an adverse pressure gradient exists, suitable control of the boundary layer is a major design consideration. This is especially true for an inlet such as that illustrated in Figure 1, since a number of oblique shock wave-boundary layer interactions occur. In a mixed-compression inlet, it is not unusual to remove 10 percent or more of the cowl lip mass flow rate by boundary layer bleed to control separation of the boundary layer.

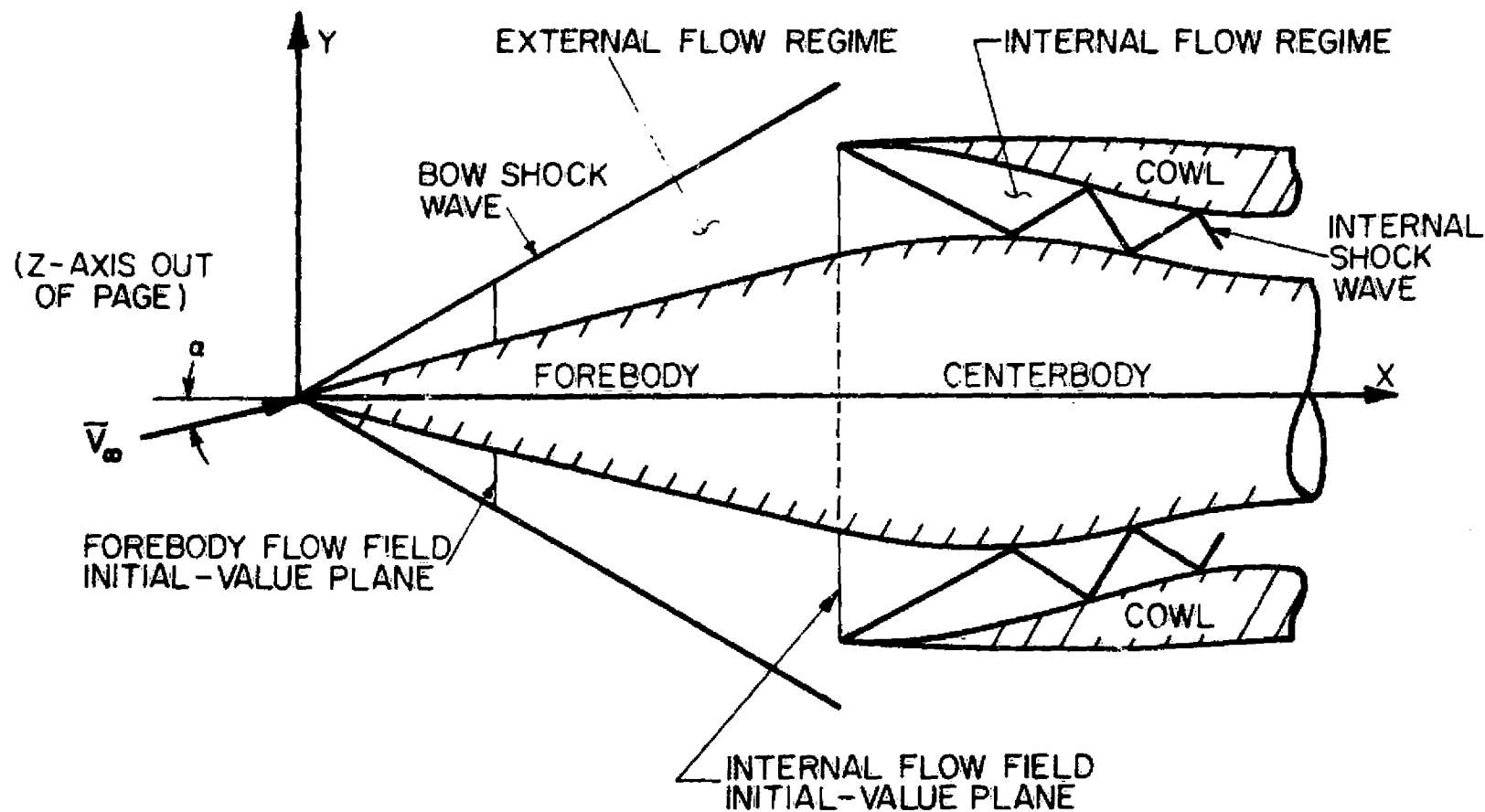


FIGURE 1. MIXED-COMPRESSION AIRCRAFT INLET

The inlet illustrated in Figure 1 is axisymmetric. At zero incidence, the flow field is axisymmetric and can be computed using a two-dimensional method. However, at nonzero angle of attack, cross flow develops, and computation of the flow field requires using a three-dimensional algorithm.

2. METHODS OF SOLUTION

The equations of motion for steady three-dimensional supersonic flow may be classified as a system of hyperbolic quasi-linear partial differential equations of first order. Exact solutions can be found in only a few cases. Consequently, most solutions are obtained by employing numerical techniques. The two most widely used numerical methods are:

1. method of finite differences
2. method of characteristics

The method of finite differences replaces the derivatives in the system of original differential equations with simple differences. The system of difference equations is then solved to obtain the solution. Finite difference methods may be further classified into those methods which do and those methods which do not contain artificial viscosity terms. The artificial viscosity terms are used to induce numerical damping and thereby reduce oscillation of the solution in regions of high flow compression. The method of characteristics first transforms the system of governing equations into characteristic form, after which the derivatives in the resulting equations are replaced by finite differences. The system of difference equations is then solved to obtain the solution. The advantages and disadvantages of each of

these methods have been discussed by Strom (1), Sauerwein (2), Richtmyer and Morton (3), and Forsythe and Wasow (4). Summarizing their findings, the finite difference methods are conceptually simpler, less difficult to program, require less computer storage, and can obtain the solution on an evenly spaced grid. The characteristics methods are, generally speaking, more accurate due to their more rigorous treatment of the physics of the problem.

In the present investigation, the flow field is computed using the method of characteristics for steady three-dimensional flow. The bow shock wave and the internal shock wave system are computed using a three-dimensional discrete shock wave fitting procedure. Moreover, the influence of molecular transport may be included in the computation by treating the viscous and thermal diffusion terms in the governing equations of motion as forcing functions, or correction terms, in the method of characteristics scheme. The primary purpose in including the effects of molecular transport in the computation is for the possible future matching of the present analysis with a higher-order boundary layer analysis. No attempt was made in the present investigation to compute the boundary layer, or to account for boundary layer removal.

3. GENERAL FEATURES OF THE THREE-DIMENSIONAL METHOD OF CHARACTERISTICS

Extensive literature surveys of the method of characteristics for three-dimensional flow have been given by Zucrow and Hoffman (5), Fowell (6), Thompson (7), Chushkin (8), Strom (1), Sauerwein (2), and Ransom, Hoffman, and Thompson (9). A brief summary of their conclusions is given here.

In general, characteristics schemes for steady three-dimensional flow may be classified as either reference plane methods or bicharacteristic methods. In reference plane methods, the system of governing partial differential equations in three-independent variables is reduced to a system of partial differential equations in two independent variables by suitably approximating the derivatives with respect to the third independent variable. These approximations to the derivatives are then treated as forcing terms, and the resulting system of equations is solved using a two-dimensional characteristics scheme. Reference plane methods have been proposed by Ferrari (10), Sauer (11,12), Ferri (13), Moretti, et al. (14,15), Katskova and Chushkin (16), Holt (17,18), and Rakich (19). Reference plane methods have been called the method of bycharacteristics by Moretti, et al. (14,15), the method of near characteristics by Sauer (11), the method of secondary characteristics by Sauer (12), and the method of semi-characteristics by Chushkin (8). In bicharacteristic methods, the characteristic equations are solved along the actual generators (bicharacteristics) of the Mach conoid and along the streamlines. Bicharacteristic schemes have been proposed by Thornhill (20), Fowell (6), Sauerwein (2,21), Coburn and Dolph (22), Holt (23), Strom (1), Butler (24), and Cline and Hoffman (25).

Reference plane methods are conceptually the simpler of the two schemes. However, reference plane methods have questionable accuracy in highly three-dimensional flows since the domain of dependence of the differential equations is not rigorously considered. Alternatively, while the bicharacteristic methods more rigorously treat the domain of dependence, they are also more complicated. The bicharacteristic

methods are potentially the more accurate and, therefore, a bicharacteristic method was selected for use in the present investigation. The particular bicharacteristic method selected was that devised by D.S. Butler (24). Butler's scheme has been applied by Elliott (26), Richardson (27), and Delaney (28) to compute unsteady two-dimensional flows. Ransom, Hoffman, and Thompson (9) applied Butler's method to compute the continuous steady three-dimensional supersonic isentropic flow field in nozzles, and Cline and Hoffman (25) applied Butler's method to compute the continuous steady three-dimensional supersonic flow field in nozzles accounting for nonequilibrium chemical reactions.

A detailed description of the computer program developed for this calculation is given in NASA TM-78947, "A Computer Program for the Calculation of the Flow Field in Supersonic Mixed-Compression Inlets at Angle of Attack Using the Three-Dimensional Method of Characteristics with Discrete Shock Wave Fitting" by Joseph Vadyak, Joe D. Hoffman, and Allan R. Bishop.

SECTION II

GAS DYNAMIC MODEL

1. INTRODUCTION

The gas dynamic model is based on the following major assumptions:

1. continuum flow
2. steady flow
3. negligible body forces
4. the working gas can be represented as a simple system in thermodynamic equilibrium
5. no mass diffusion
6. negligible radiative heat transfer and no internal heat generation other than viscous dissipation
7. viscous and thermal diffusion effects of secondary importance

The governing equations for the assumed flow model consist of the continuity equation, the component momentum equations, the energy equation, the thermal and caloric equations of state, and appropriate representations for the molecular transport properties. These equations are briefly presented in this section. A detailed development of these equations is given in Appendix A.

2. GOVERNING DIFFERENTIAL EQUATIONS OF MOTION

The continuity equation* [see Reference (29)] is given by

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \quad (1)$$

where x_i ($i=1,2,3$) denotes the three rectangular cartesian coordinates x , y , and z , respectively, u_i ($i=1,2,3$) denotes the corresponding velocity components u , v , and w , respectively, ρ denotes the density, and t denotes the time. The operator $D(\)/Dt$ in equation (1) is the material derivative given by

$$\frac{D(\)}{Dt} = \frac{\partial(\)}{\partial t} + u_j \frac{\partial(\)}{\partial x_j} \quad (2)$$

For steady three-dimensional flow, equation (1) may be written in expanded form as

$$\rho u_x + \rho v_y + \rho w_z + u\rho_x + v\rho_y + w\rho_z = 0 \quad (3)$$

where the subscripts x , y , and z denote partial differentiation with respect to the corresponding direction.

The momentum equation is given by the Navier-Stokes equation (29), which written in component form is

$$\begin{aligned} \rho \frac{Du_i}{Dt} = B_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \\ - \frac{2}{3} \frac{\partial}{\partial x_i} \left(\mu \frac{\partial u_j}{\partial x_j} \right) + \frac{\partial}{\partial x_i} \left(\eta \frac{\partial u_j}{\partial x_j} \right) \quad (i=1,2,3) \end{aligned} \quad (4)$$

* Repeated indices imply summation over the range of 1 to 3 unless otherwise noted.

where B_i ($i=1,2,3$) denotes the x , y , and z components of the body force, respectively, P denotes the pressure, μ denotes the dynamic viscosity, and η denotes the second coefficient of viscosity.

One of the major assumptions of the present investigation is that the influence of molecular transport is considered to be of secondary importance as compared to the inertial effects in determining the solution. As a consequence, the viscous and thermal diffusion terms appearing in the governing partial differential equations are treated as forcing functions or correction terms in the method of characteristics scheme to be presented. In what follows, the molecular transport terms are placed on the right-hand sides of the respective governing equations, and the convective terms are placed on the left-hand sides of those equations. The convective terms then are considered as constituting the principal parts of these equations. Hence, by assuming steady flow, negligible body forces, $\eta = 0$ [Stokes's hypothesis (30)], inertial dominance, and variable transport properties, equation (4) may be written in expanded form for each of the three coordinate directions as

$$\rho u u_x + \rho v u_y + \rho w u_z + P_x = F_x \quad (5)$$

$$\rho u v_x + \rho v v_y + \rho w v_z + P_y = F_y \quad (6)$$

$$\rho u w_x + \rho v w_y + \rho w w_z + P_z = F_z \quad (7)$$

where

$$F_x = \mu_x \left[\frac{4}{3} u_x - \frac{2}{3} (v_y + w_z) \right] + \mu_y (u_y + v_x) + \mu_z (u_z + w_x) + \mu \left[\frac{4}{3} u_{xx} + u_{yy} + u_{zz} + \frac{1}{3} (v_{xy} + w_{xz}) \right] \quad (8)$$

$$F_y = \mu_y \left[\frac{4}{3} v_y - \frac{2}{3} (u_x + w_z) \right] + \mu_x (v_x + u_y) + \mu_z (v_z + w_y) \\ + \mu \left[\frac{4}{3} v_{yy} + v_{xx} + v_{zz} + \frac{1}{3} (u_{yx} + w_{yz}) \right] \quad (9)$$

$$F_z = \mu_z \left[\frac{4}{3} w_z - \frac{2}{3} (u_x + v_y) \right] + \mu_x (w_x + u_z) + \mu_y (w_y + v_z) \\ + \mu \left[\frac{4}{3} w_{zz} + w_{xx} + w_{yy} + \frac{1}{3} (u_{zx} + v_{zy}) \right] \quad (10)$$

The appropriate form of the energy equation is now derived. In the following, the pressure P and density ρ are considered as being the primary thermodynamic variables. All secondary thermodynamic variables are then expressed in terms of the pressure and density.

It is assumed in the present investigation that the working gas may be represented as a simple system in thermodynamic equilibrium. For a simple system, specification of any two independent thermodynamic properties defines the thermodynamic state of the system (31). Hence, the following functional relationship may be written

$$P = P(\rho, s) \quad (11)$$

where s is the entropy per unit mass. Employing the concept of the total derivative, and introducing the material derivative operator given by equation (2), the following equation is obtained.

$$\frac{DP}{Dt} = \left(\frac{\partial P}{\partial \rho} \right)_s \frac{D\rho}{Dt} + \left(\frac{\partial P}{\partial s} \right)_\rho \frac{Ds}{Dt} \quad (12)$$

The sonic speed a is defined by

$$a^2 = \left(\frac{\partial P}{\partial \rho} \right)_s \quad (13)$$

Introducing equation (13) into equation (12) yields

$$\frac{DP}{Dt} - a^2 \frac{D\rho}{Dt} = \left(\frac{\partial P}{\partial s} \right)_\rho \frac{Ds}{Dt} \quad (14)$$

The material derivative of entropy appearing in equation (14) may be expressed in terms of a thermal conduction function and a viscous dissipation function. The entropy may be expressed in terms of the internal energy by use of the thermodynamic relation (31)

$$T ds = de + P d(1/\rho) \quad (15)$$

where T is the temperature, and e is the internal energy per unit mass. The internal energy may be expressed in terms of a thermal conduction function and a viscous dissipation function by use of the energy equation (29)

$$\rho \frac{De}{Dt} = \frac{\partial}{\partial x_i} \left[\kappa \frac{\partial T}{\partial x_i} \right] + \frac{P}{\rho} \frac{D\rho}{Dt} + \phi \quad (16)$$

where κ is the thermal conductivity, and ϕ is the viscous dissipation function, which for $\eta = 0$ is given by

$$\phi = \frac{1}{2} \mu \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right]^2 \quad (17)$$

where δ_{ij} is the Kronecker delta. Combining equations (14) to (17) and writing the resulting expression in expanded form for steady three-dimensional flow with variable transport properties yields

$$uP_x + vP_y + wP_z - a^2(u\rho_x + v\rho_y + w\rho_z) = F_e \quad (18)$$

where

$$F_e = \xi \left\{ \kappa(T_{xx} + T_{yy} + T_{zz}) + \kappa_x T_x + \kappa_y T_y + \kappa_z T_z + \mu[2(u_x^2 + v_y^2 + w_z^2 + u_y v_x + u_z w_x + v_z w_y) + v_x^2 + w_x^2 + u_y^2 + w_y^2 + u_z^2 + v_z^2 - \frac{2}{3}(u_x + v_y + w_z)^2] \right\} \quad (19)$$

and

$$\xi = \frac{1}{\rho T} \left(\frac{\partial P}{\partial s} \right)_\rho \quad (20)$$

3. THERMODYNAMIC MODEL

Before a solution to the system of governing partial differential equations may be obtained, the temperature T , the sonic speed a , and the parameter ξ defined by equation (20) must be expressed in terms of the primary thermodynamic variables P and ρ . The general functional forms of the relations for T , a , and ξ are given by

$$T = T(P, \rho) \quad (21)$$

$$a = a(P, \rho) \quad (22)$$

$$\xi = \xi(P, \rho) \quad (23)$$

The derivatives of the temperature appearing in equation (19) are expressed in terms of the derivatives of the pressure and the density by analytically differentiating equation (21).

For the special case of a thermally and calorically perfect gas, equations (21) to (23) take the following simple forms

$$T = P/\rho R \quad (24)$$

$$a = (\gamma P/\rho)^{1/2} \quad (25)$$

$$\xi = \gamma - 1 \quad (26)$$

where R is the gas constant, and γ is the specific heat ratio.

4. MOLECULAR TRANSPORT PROPERTIES

The dynamic viscosity μ and the thermal conductivity κ must be expressed in terms of the primary thermodynamic variables P and ρ . In general, both the viscosity and the thermal conductivity are assumed to be functions of temperature only. Hence,

$$\mu = \mu(T) \quad (27)$$

$$\kappa = \kappa(T) \quad (28)$$

The derivatives of the transport properties appearing in equations (8), (9), (10), and (19) are obtained in terms of the derivatives of the pressure and the density by analytically differentiating equations (27) and (28) with respect to the temperature, with the resulting temperature derivatives being obtained by analytically differentiating equation (21).

A widely accepted representation for equation (27) is the Sutherland formula (30)

$$\mu = \mu_0 \left(\frac{T}{T_0} \right)^{1.5} \left[\frac{T_0 + S}{T + S} \right] \quad (29)$$

where μ_0 is the viscosity at the reference temperature T_0 , and S is a constant. Equation (28) may be represented by the quadratic polynomial

$$\kappa = a_1 + a_2 T + a_3 T^2 \quad (30)$$

where the coefficients a_i ($i=1,2,3$) are obtained by curve fitting thermal conductivity data.

The contribution of turbulent transport may be considered in the computation by adding the appropriate eddy viscosity and eddy thermal conductivity functions to the molecular transport properties given by equations (27) and (28), respectively.

5. SUMMARY

In summary, the differential equations of motion for steady three-dimensional flow are given by equations (3), (5), (6), (7), and (18). For a thermally and calorically perfect gas, the thermodynamic model is represented by equations (24) to (26). The molecular transport properties are represented by equations (29) and (30).

SECTION III

CHARACTERISTIC EQUATIONS

1. INTRODUCTION

Written in the form shown, with the left-hand sides constituting the principal parts, equations (3), (5), (6), (7), and (18) may be classified as a system of quasi-linear nonhomogenous partial differential equations of first order. The system is hyperbolic if the flow is supersonic. Systems of hyperbolic partial differential equations in three independent variables have the property that there exist surfaces in three-dimensional space on which linear combinations of the original partial differential equations can be formed that contain derivatives only in the surfaces themselves. These special surfaces are known as characteristic surfaces, and the linear combinations of the original partial differential equations are interior differential operators known as compatibility relations. In this section, the equations for the characteristic surfaces and the compatibility relations valid along these surfaces are listed and briefly discussed. A detailed development of these equations is given in Appendix B.

2. CHARACTERISTIC SURFACES

For steady three-dimensional supersonic flow, two families of characteristic surfaces exist, as illustrated in Figure 2. One family of characteristic surfaces consists of the stream surfaces given by

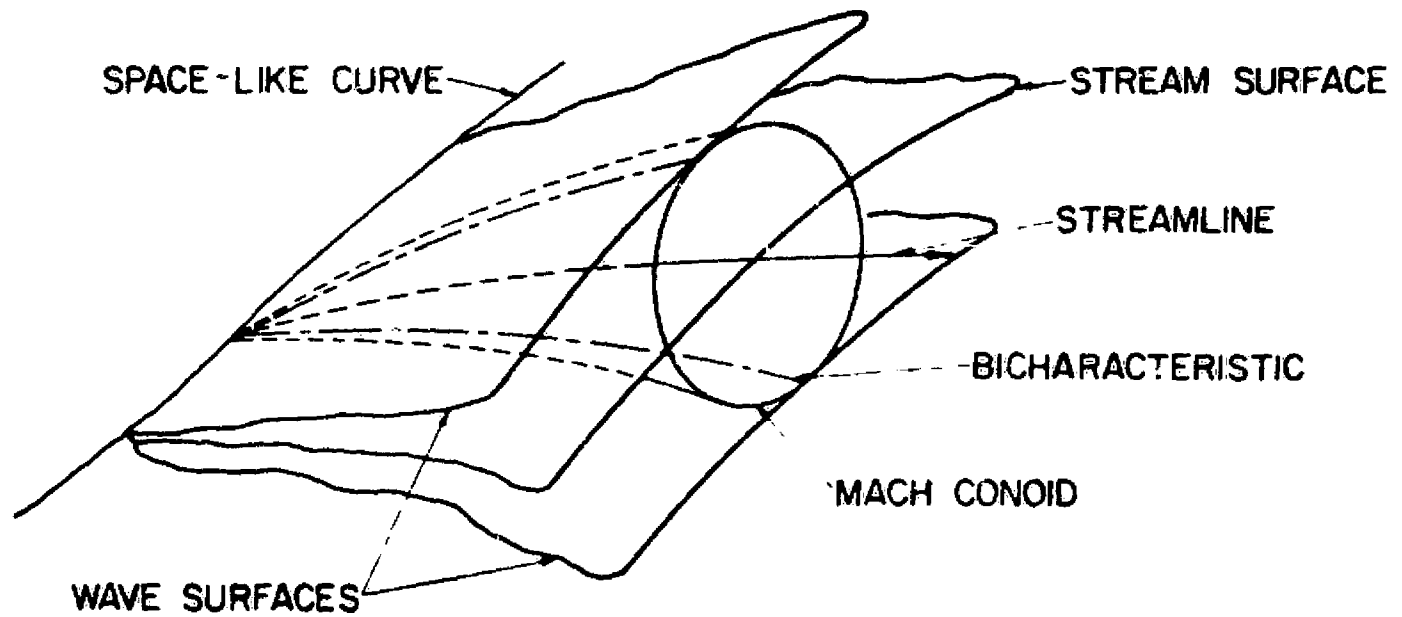


FIGURE 2. CHARACTERISTIC SURFACES

$$uN_x + vN_y + wN_z = 0 \quad (31)$$

where $\bar{N} = (N_x, N_y, N_z)$ denotes the normal to a stream surface. The envelope of all stream surfaces at a point forms a single pencil of planes whose axis is a streamline. A streamline may be represented by

$$dx/dt = u \quad dy/dt = v \quad dz/dt = w \quad (32)$$

where t is the time of travel of a fluid particle along the streamline.

The second family of characteristic surfaces consists of the wave surfaces given by

$$uN_x + vN_y + wN_z = a|\bar{N}| \quad (33)$$

where $\bar{N} = (N_x, N_y, N_z)$ denotes the normal to a wave surface. The envelope of all wave surfaces at a point forms a conoid known as the Mach conoid. The Mach conoid may be represented locally by a right circular cone known as the Mach cone. In differential form, the quadric surface of the Mach conoid is given by

$$\begin{aligned} &[u^2 - (q^2 - a^2)](dx)^2 + [v^2 - (q^2 - a^2)](dy)^2 \\ &+ [w^2 - (q^2 - a^2)](dz)^2 + 2uv(dx)(dy) \\ &+ 2uw(dx)(dz) + 2vw(dy)(dz) = 0 \end{aligned} \quad (34)$$

where q is the velocity magnitude ($q^2 = u^2 + v^2 + w^2$). The line of contact between a particular wave surface and the Mach conoid is known as a bicharacteristic. A bicharacteristic is a generator of the Mach conoid.

3. COMPATIBILITY RELATIONS

The compatibility relations which are applicable on the stream surfaces are given by

$$uP_x + vP_y + wP_z - a^2(u\rho_x + v\rho_y + w\rho_z) = F_e \quad (35)$$

$$\begin{aligned} & \rho u(uu_x + vu_y + wu_z) + \rho v(uv_x + vv_y + wv_z) \\ & + \rho w(uw_x + vw_y + ww_z) + uP_x + vP_y + wP_z \\ & = uF_x + vF_y + wF_z \end{aligned} \quad (36)$$

$$\begin{aligned} & \rho S_x(uu_x + vu_y + wu_z) + \rho S_y(uv_x + vv_y + wv_z) \\ & + \rho S_z(uw_x + vw_y + ww_z) + S_x P_x + S_y P_y + S_z P_z \\ & = S_x F_x + S_y F_y + S_z F_z \end{aligned} \quad (37)$$

In equation (37), $\bar{S} = (S_x, S_y, S_z)$ denotes a vector which lies in the stream surface and that is independent of the velocity vector. Equations (35) and (36) may be written in a form which contains differentiation in the streamline direction as follows.

$$\frac{dP}{dt} - a^2 \frac{d\rho}{dt} = F_e \quad (38)$$

$$\rho u \frac{du}{dt} + \rho v \frac{dv}{dt} + \rho w \frac{dw}{dt} + \frac{dP}{dt} = uF_x + vF_y + wF_z \quad (39)$$

In equations (38) and (39), the operator $d(\)/dt$ represents the directional derivative along a streamline.

The compatibility relation which is applicable on the wave surfaces is given by

$$\begin{aligned} & \rho a n_x (u u_x + v u_y + w u_z) + \rho a n_y (u v_x + v v_y + w v_z) \\ & + \rho a n_z (u w_x + v w_y + w w_z) + (a n_x - u) P_x + (a n_y - v) P_y \\ & + (a n_z - w) P_z - \rho a^2 (u_x + v_y + w_z) = \lambda \end{aligned} \quad (40)$$

where

$$\lambda = a(n_x F_x + n_y F_y + n_z F_z) - F_e \quad (41)$$

In equations (40) and (41), $\vec{n} = (n_x, n_y, n_z)$ denotes the unit normal vector to the wave surface. Equation (40) may be written in a form which contains differentiation in the bicharacteristic direction as follows.

$$\begin{aligned} & \rho a n_x \frac{du}{dt} + \rho a n_y \frac{dv}{dt} + \rho a n_z \frac{dw}{dt} - \frac{dP}{dt} = \lambda - \rho a^2 [(n_x^2 - 1)u_x \\ & + (n_y^2 - 1)v_y + (n_z^2 - 1)w_z + n_x n_y (u_y + v_x) \\ & + n_x n_z (u_z + w_x) + n_y n_z (v_z + w_y)] \end{aligned} \quad (42)$$

In equation (42), the operator $d(\)/dt$ denotes the directional derivative along a bicharacteristic. The terms in brackets in equation (42) represent differentiation in the wave surface but in a direction normal to the bicharacteristic direction. Hereafter, these terms will be referred to as the cross derivatives.

At any point in the flow field there exists an infinite number of stream surfaces and wave surfaces. The number of independent

compatibility relations cannot exceed the number of independent equations of motion. As a consequence, it is necessary to determine which of the possible combinations of the compatibility relations form an independent set. Rusanov (32), using a proof in the space of characteristic normals, has shown for steady three-dimensional isentropic flow that two of the stream surface compatibility relations applied along a stream surface and the single wave surface compatibility relation applied along three different wave surfaces form an independent set of five characteristic relations. Rusanov's results may be extended to the present case since the principal parts of equations (3), (5), (6), (7), and (18) are the same as those for isentropic flow. Hence, the set of compatibility relations used in the present investigation consists of equations (38) and (39) applied along a streamline and equation (42) applied along three different bicharacteristics.

4. BUTLER'S PARAMETERIZATION OF THE CHARACTERISTIC EQUATIONS

D. S. Butler (24) developed a parameteric form for representing a bicharacteristic and the wave surface compatibility relation applicable along it. A detailed development of Butler's method is presented in Appendix B. A brief summary is given here.

Butler introduced the following parameteric form to represent a bicharacteristic.

$$dx_i = (u_i + c\alpha_i \cos\theta + c\beta_i \sin\theta)dt \quad (i=1,2,3) \quad (43)$$

In equation (43), t is the time of travel of a fluid particle along the streamline that is the axis of the Mach cone, θ is a parametric

angle denoting a particular element of the Mach cone and has the range $0 \leq \theta \leq 2\pi$, and c is given by

$$c^2 = q^2 a^2 / (q^2 - a^2) \quad (44)$$

where q is the velocity magnitude, and a is the sonic speed. The vectors α_i and β_i in equation (43) are parametric unit vectors with α_i , β_i , and u_i/q ($i=1,2,3$) forming an orthonormal set.

The corresponding parametric form of the wave surface compatibility relation, equation (40), is given by

$$\begin{aligned} \frac{dP}{dt} + \rho c (\alpha_i \cos \theta + \beta_i \sin \theta) \frac{du_i}{dt} = \Phi \\ - \rho c^2 (\alpha_i \sin \theta - \beta_i \cos \theta) (\alpha_j \sin \theta - \beta_j \cos \theta) \frac{\partial u_i}{\partial x_j} \end{aligned} \quad (45)$$

In equation (45), the operator $d(\)/dt$ represents differentiation in the bicharacteristic direction, and Φ is given by

$$\Phi = (c^2/a^2) [F_e - a(n_x F_x + n_y F_y + n_z F_z)] \quad (46)$$

where $\hat{n} = (n_x, n_y, n_z)$ denotes the unit normal to the wave surface, which may be written in parametric form as

$$n_i = (a/c)(cu_i/q^2 - \alpha_i \cos \theta - \beta_i \sin \theta) \quad (i=1,2,3) \quad (47)$$

In addition to the above relations, Butler also developed a non-characteristic relation which is applied along a streamline. This noncharacteristic relation is given by

$$\frac{dP}{dt} = \sigma - \rho c^2 (\alpha_i \alpha_j + \beta_i \beta_j) \frac{\partial u_j}{\partial x_j} \quad (48)$$

where the operator $d()/dt$ denotes differentiation along a streamline,
and σ is given by

$$\sigma = (c^2/a^2)F_e - (c^2/q^2)(uF_x + vF_y + wF_z) \quad (49)$$

SECTION IV

UNIT PROCESSES

1. INTRODUCTION

A variety of unit processes are employed in the computation of the flow field. The unit processes may be classified into four major types: interior point, solid boundary point, field-shock wave point, and solid body-shock wave point. The basic unit processes are briefly discussed in this section. A detailed presentation of each unit process is given in Appendix E.

In the overall numerical algorithm, an inverse marching scheme is employed. The solution is obtained on space-like planes of constant x , where the x -axis is the longitudinal axis of the centerbody and the cowl. For the internal flow, the solution is also obtained on the space curves which are defined by the intersections of the internal shock wave with the solid boundaries. Except in the vicinity of a shock wave-solid boundary intersection, the distance Δx between successive solution planes is determined by the application of the Courant-Friedrichs-Lewy (CFL) stability criterion (9). In the vicinity of a shock wave-solid boundary intersection, the axial step is controlled by special constraints, which are discussed in Section V. The distance Δx is determined prior to the application of the unit processes.

2. INTERIOR POINT UNIT PROCESS

The computational network used in determining the solution for a typical interior point is illustrated in Figure 3. Points (1) to (4) represent the intersection points of four rearward-running bicharacteristics with the initial-value plane, point (5) is the streamline intersection point with the initial-value plane, and point (6) is the solution point on the solution plane. The axial (x) distance between the initial-value plane and the solution plane is determined prior to the application of the unit process by applying the CFL stability criterion. As in all the unit processes, the interior point unit process is divided into a predictor step and a corrector step. The corrector may be iterated to convergence if desired.

The interior point unit process is initiated by determining the location of the solution point, point (6). The coordinates of point (6) are determined by extending the streamline from point (5) to the solution plane using the following finite difference form of equation (32).

$$x_i(6) - x_i(5) = \frac{1}{2} [u_i(5) + u_i(6)][t(6) - t(5)] \quad (i=1,2,3) \quad (50)$$

For the predictor, $u_i(6)$ is equated to $u_i(5)$. For the corrector, the previously determined value of $u_i(6)$ is employed. The axial step $[x(6) - x(5)]$ is computed before the unit process is applied. Hence, the time parameter $[t(6) - t(5)]$ may be obtained, after which the coordinates $y(6)$ and $z(6)$ are computed. Interpolated flow property values at point (5) are used in the integration, even though point (5) is a known field point. As shown by Ransom, Hoffman, and Thompson (9),

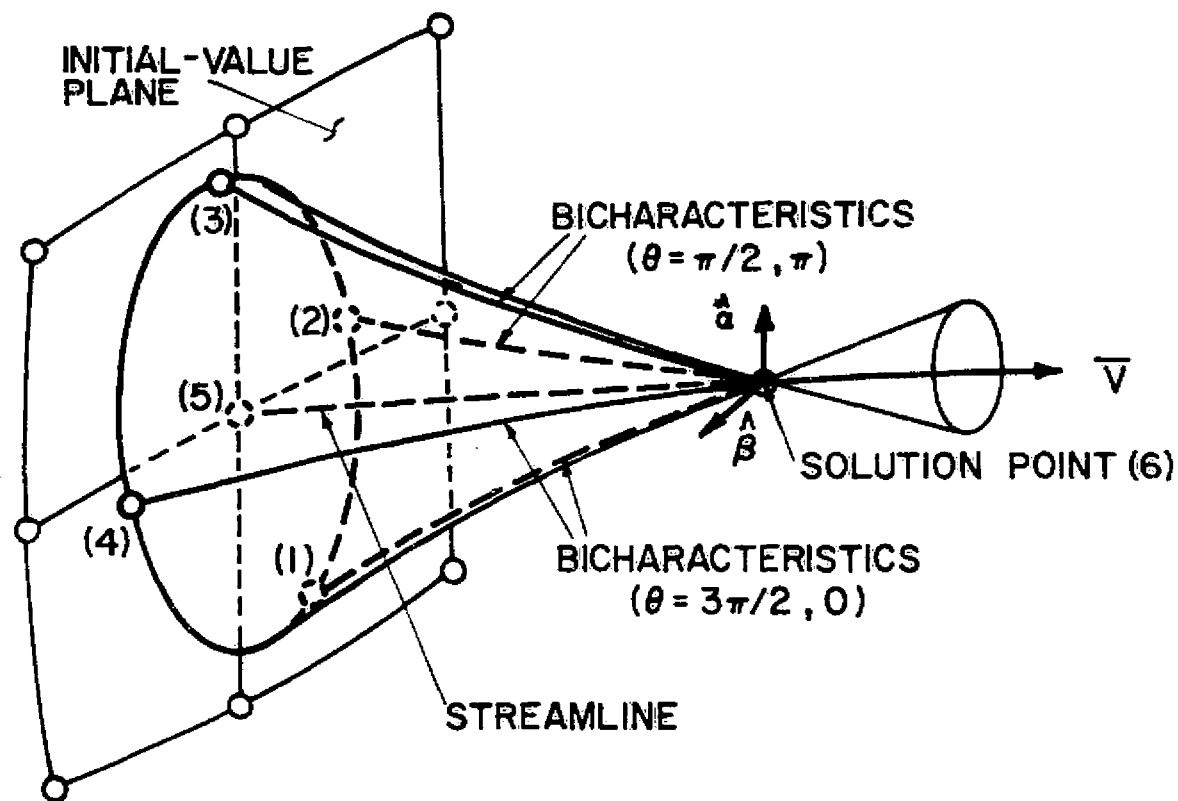


FIGURE 3. INTERIOR POINT COMPUTATIONAL NETWORK

this interpolation is required to produce a stable numerical scheme. The interpolated flow property values are obtained from the following quadratic bivariate interpolation polynomial

$$f(y,z) = a_1 + a_2y + a_3z + a_4yz + a_5y^2 + a_6z^2 \quad (51)$$

where $f(y,z)$ denotes a general function of the coordinates y and z , and the coefficients a_i ($i=1$ to 6) are obtained from a least squares fit of nine data points in the initial-value plane [point (5) and its eight immediate neighbors] as described in Appendix C.

With the location of the solution point determined, four bicharacteristics are extended from the solution point back to the initial-value plane to intersect this plane at points (1) to (4), as illustrated in Figure 3. The coordinates of each of these intersection points are determined using the following finite difference form of equation (43).

$$\begin{aligned} x_i(6) - x_i(k) = \frac{1}{2} \{u_i(k) + u_i(6) + [c(k) + c(6)][\alpha_i \cos \theta(k) \\ + \beta_i \sin \theta(k)]\} [t(6) - t(k)] \quad (i=1,2,3) \end{aligned} \quad (52)$$

The index k in equation (52) denotes the bicharacteristic-initial-value plane intersection points illustrated in Figure 3, and has a range of 1 to 4, corresponding to the $\theta(k)$ values of 0 , $\pi/2$, π , and $3\pi/2$, respectively. Since the axial step $[x(6) - x(k)]$ is known, equation (52) is used to calculate $[t(6) - t(k)]$, $y(k)$, and $z(k)$. The flow property values at points (1) to (4) are obtained by interpolation using equation (51). On the initial application of equation (52), the flow property values at point (k) are equated to those at point (5).

For the external flow field integration, the parametric unit vectors α_i and β_i appearing in equation (52) are selected to straddle the projection of the pressure gradient on the initial-value plane. For the internal flow field integration, these vectors are selected to straddle the meridional plane through point (6).

Once the positions of and the flow properties at points (1) to (5) have been determined, the system of nonlinear compatibility equations, written in finite difference form, is solved to obtain the five dependent flow properties $u(6)$, $v(6)$, $w(6)$, $P(6)$ and $\rho(6)$. Two of the five required compatibility equations are given by equations (38) and (39). These equations are written in finite difference form by replacing the derivatives with simple differences, and by replacing the coefficients of the derivatives with the arithmetic average of the coefficients at the solution point and at the appropriate point in the initial-value plane. To obtain the remaining three required compatibility equations, appropriate linear combinations of the wave surface compatibility relation, equation (45), applied along each of the four bicharacteristics, and the noncharacteristic relation, equation (48), applied along the streamline are formed. Writing equation (45) for θ values of 0 , $\pi/2$, π , and $3\pi/2$ yields

$$\frac{dP}{dt_1} + \rho c \alpha_i \frac{du_i}{dt_1} = \phi_1 - \rho c^2 \beta_i \beta_j \frac{\partial u_i}{\partial x_j} \quad (53)$$

$$\frac{dP}{dt_2} + \rho c \beta_i \frac{du_i}{dt_2} = \phi_2 - \rho c^2 \alpha_i \alpha_j \frac{\partial u_i}{\partial x_j} \quad (54)$$

$$\frac{dp}{dt_3} - \rho c \alpha_i \frac{du_i}{dt_3} = \phi_3 - \rho c^2 \beta_i \beta_j \frac{\partial u_i}{\partial x_j} \quad (55)$$

$$\frac{dp}{dt_4} - \rho c \beta_i \frac{du_i}{dt_4} = \phi_4 - \rho c^2 \alpha_i \alpha_j \frac{\partial u_i}{\partial x_j} \quad (56)$$

In equations (53) to (56), the operator $d()/dt_k$ denotes differentiation along the k th bicharacteristic, and ϕ_k denotes equation (46) evaluated for the specified value of $\theta(k)$. One independent linear combination of the compatibility equations is obtained by subtracting the finite difference form of equation (55) from the finite difference form of equation (53). Another independent linear combination is obtained by subtracting the finite difference form of equation (56) from the finite form of equation (54). The final independent linear combination is obtained by subtracting the finite difference form of the noncharacteristic relation, equation (48), from the sum of the finite difference forms of equations (53) and (54). The resulting compatibility equations do not contain cross derivatives at the solution point [i.e., all terms containing $\partial u_i / \partial x_j$ (6) are eliminated]. These five finite difference equations are solved using Gaussian elimination. For the predictor, the flow property values at the solution point appearing in the coefficients of the derivatives in the set of difference equations are equated to those at point (5). For the corrector, the flow property values at point (6) obtained on the previous iteration are used. The resulting scheme has second-order accuracy (9).

3. SOLID BOUNDARY POINT UNIT PROCESS

The computational network used for determining the solution at a typical point on a solid boundary is shown in Figure 4. The point notation used in this figure is identical to that employed in Figure 3. Here, however, both points (5) and (6) lie on the solid boundary, and point (4) is not used since it lies outside of the flow regime.

The unit process used to obtain the solution at a solid boundary point is almost identical to the interior point unit process. Here, however, point (4) corresponding to the bicharacteristic with $\theta = 3\pi/2$ is not located, and the corresponding compatibility relation valid along this bicharacteristic is not employed. That equation is replaced by the flow tangency condition

$$u_i(6)n_{bi}(6) = 0 \quad (57)$$

where $n_{bi}(6)$ ($i=1,2,3$) is the unit normal to the solid boundary at point (6).

4. BOW SHOCK WAVE POINT UNIT PROCESS

The computational network used in determining the solution for a typical bow shock wave point is illustrated in Figure 5. A segment of the shock wave surface extending from the initial-value plane to the solution plane is shown in this figure. The intersection of the shock wave with the initial-value plane defines space curve (A), and the intersection of the shock wave with the solution plane defines space curve (B). The axial distance between the initial-value plane and the solution plane has been previously determined by the application of the CFL stability criterion. The bow shock wave solution point

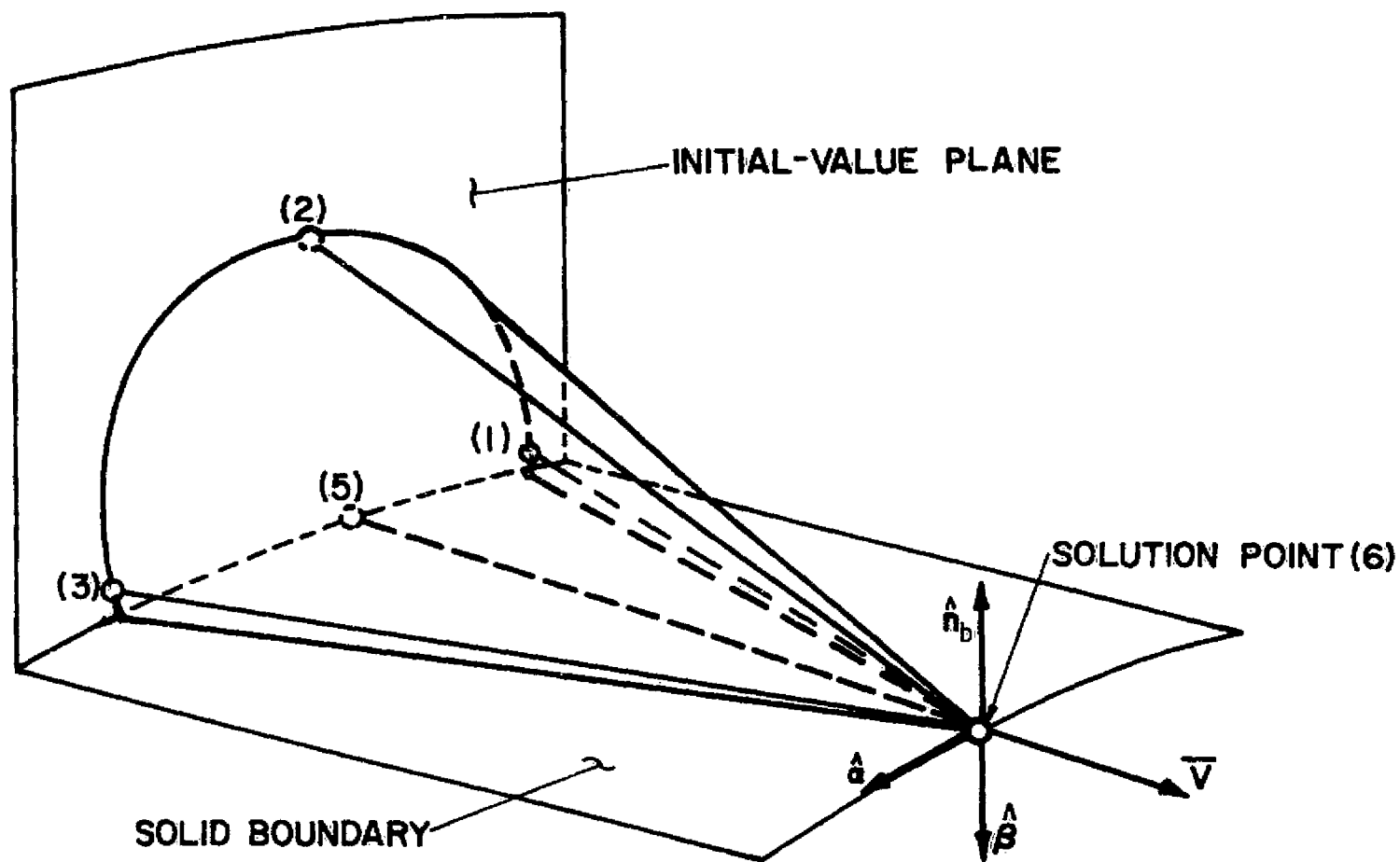


FIGURE 4. SOLID BOUNDARY POINT COMPUTATIONAL NETWORK

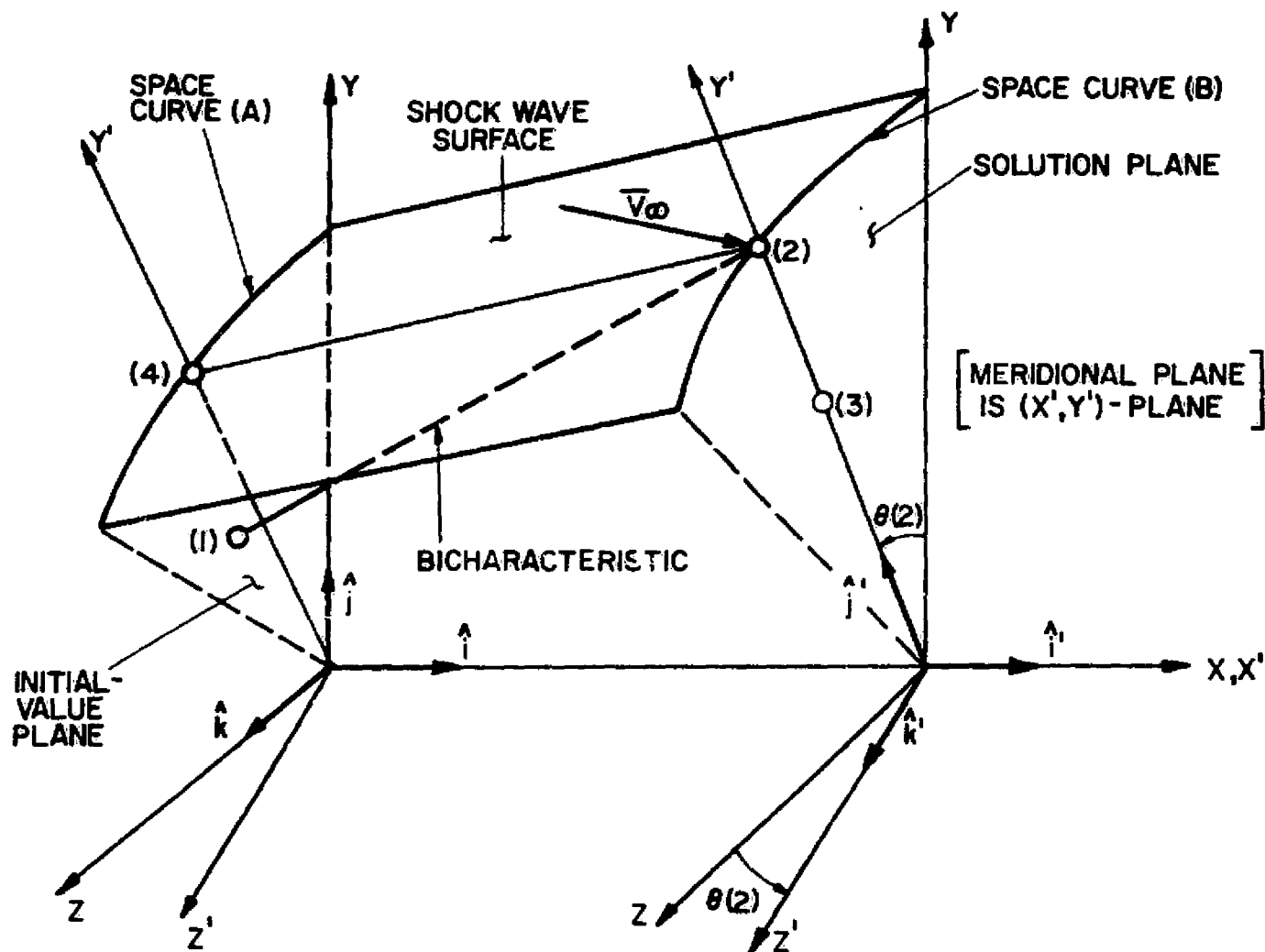


FIGURE 5. BOW SHOCK WAVE POINT COMPUTATIONAL NETWORK

is denoted by point (2). The flow properties at point (2) on the upstream side of the shock wave are known from the free-stream conditions. Hence, in the following discussion, the flow properties $u(2)$, $v(2)$, $w(2)$, $P(2)$, and $\rho(2)$ refer to the flow properties at point (2) on the downstream side of the shock wave. Point (1) is the intersection point of a rearward-running bicharacteristic with the initial-value plane. This bicharacteristic is extended backward from the solution point, point (2). Point (3) is a predetermined interior solution point which is adjacent to the shock wave and is used to define the meridional plane in which the bow shock wave solution point lies. Point (4) is the intersection point of space curve (A) with the meridional plane which passes through points (2) and (3).

In this unit process, a local cartesian coordinate system is employed for the description of the local shock wave surface. This local coordinate system has coordinates x' , y' , and z' , where x' is coincident with the x -axis, y' is the radial direction in the meridional plane containing points (2) and (3), and z' is normal to the (x',y') -plane. The unit vectors in the x' , y' , and z' directions are denoted by $\hat{i}'$, $\hat{j}'$, and $\hat{k}'$, respectively. The orientation of the local shock wave surface at a point (P) is specified by a set of three unit vectors referenced to the (x',y',z') -coordinate system, as illustrated in Figure 6. This set of unit vectors consists of the unit vector $\hat{n}_s$ which is normal to the shock wave surface at point (P), and two unit vectors $\hat{l}$ and $\hat{t}$ which are tangent to this surface at point (P). The tangential unit vector $\hat{t}$ lies in the meridional plane $[(x',y')\text{-plane}]$, subtends an angle ϕ with the x' -axis, and is defined by the intersection

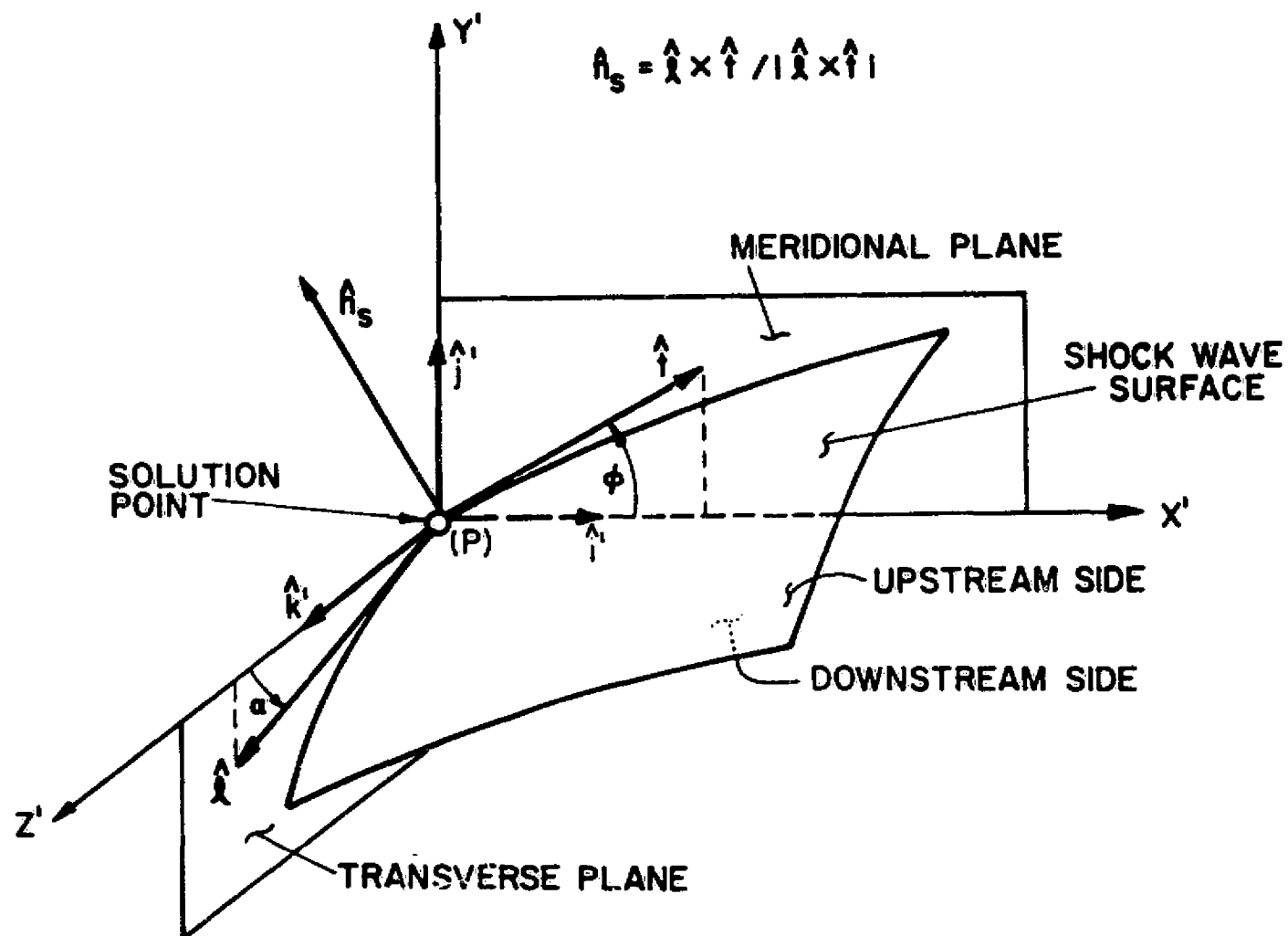


FIGURE 6. UNIT VECTORS FOR SPECIFICATION OF SHOCK WAVE SURFACE ORIENTATION

of the shock wave with the meridional plane at point (P). The tangential unit vector $\hat{\ell}$ lies in the transverse plane [(y',z')-plane], subtends an angle α with the z'-axis, and is defined by the intersection of the shock wave with the transverse plane at point (P). The tangential unit vectors $\hat{t}$ and $\hat{\ell}$ are given by

$$\hat{t} = \cos \phi \hat{i}' + \sin \phi \hat{j}' \quad (58)$$

$$\hat{\ell} = \sin \alpha \hat{j}' + \cos \alpha \hat{k}' \quad (59)$$

The shock wave normal unit vector $\hat{n}_s$ is given by

$$\hat{n}_s = \hat{\ell} \times \hat{t} / |\hat{\ell} \times \hat{t}| \quad (60)$$

To achieve second-order accuracy in the shock wave point unit process, global iteration must be performed. In global iteration, the corrector employs flow properties not only at the solution point itself, but also at neighboring points in the solution plane. As a consequence, before the corrector can be applied in global iteration, the entire solution plane (or at least an appropriate section of it) must be determined by a prior calculation. The interior point and solid boundary point unit processes do not require global iteration to achieve second-order accuracy. Consequently, those solution points are determined first. Then, the predictor is applied for each shock wave solution point, thereby giving a tentative solution for all of the shock wave points. At this stage, global correction is performed for the shock wave solution points using the previously determined field points in the solution plane. In the following discussion, the term "predictor" refers to the first application of the shock wave

point unit process used to obtain an initial estimate of the solution without using field point data in the solution plane. The term "global corrector" refers to the application of the shock wave point unit process which uses field point data in the solution plane. The shock wave point unit process is now outlined.

The shock wave point unit process is initiated by locating the solution point, point (2) in Figure 5. Denote the angle subtended by a meridional plane and the (x,y)-plane by θ . The solution point meridional plane is arbitrarily selected to contain the interior solution point, point (3), whose location is determined prior to the application of the shock wave point unit process. Hence, $\theta(2) = \theta(3)$. Denote the radial position of a point by r . Then the radial position of point (2) is obtained from

$$r(2) = r(4) + [x(2) - x(4)] \tan \left\{ \frac{1}{2} [\phi(2) + \phi(4)] \right\} \quad (61)$$

where $[x(2) - x(4)]$ is the axial distance between the initial-value plane and the solution plane. On the initial application of equation (61), the shock wave angle $\phi(2)$ is equated to $\phi(4)$, whereas, on ensuing applications, the value of $\phi(2)$ obtained on the previous iteration is used. At point (4), the radial position $r(4)$ and the shock wave angle $\phi(4)$ are determined by interpolation using the quadratic univariate formulae

$$r(\theta) = a_1 + a_2\theta + a_3\theta^2 \quad (62)$$

$$\phi(\theta) = b_1 + b_2\theta + b_3\theta^2 \quad (63)$$

where the coefficients a_i and b_i ($i=1,2,3$) are determined by fitting these expressions to three local shock wave solution points on space curve (A).

After the solution point has been located, the shock wave normal unit vector $\hat{n}_s$ at the solution point is found by forming the normalized cross product of the tangential unit vectors $\hat{l}$ and $\hat{t}$ [see equation (60)]. The tangential unit vectors $\hat{t}$ and $\hat{l}$ are obtained by use of the current values of $\phi(2)$ and $\alpha(2)$ in equations (58) and (59), respectively. For a predictor application, $\alpha(2)$ is approximated by equating it to the α value at point (4). For a global corrector application, the value of $\alpha(2)$ that is employed is that evaluated at point (2). In either case, the value of $\alpha(2)$ may be determined by

$$\alpha(2) = \tan^{-1} \left\{ \frac{1}{r} \frac{dr}{d\theta} \right\} \bigg|_{\theta(2)} \quad (64)$$

where, for the predictor, the analytical form of $r(\theta)$ used in equation (64) is given by equation (62) applied along space curve (A), and for the global corrector, $r(\theta)$ is obtained by applying equation (62) along space curve (B).

At this stage, the local Hugoniot relations are applied at point (2) to obtain the downstream flow properties $u(2)$, $v(2)$, $w(2)$, $P(2)$, and $\rho(2)$. Next, a rearward-running bicharacteristic is extended from the solution point, point (2), back to the initial-value plane, intersecting this plane at point (1), as illustrated in Figure 5. The coordinates of point (1) are obtained using the following finite difference form of equation (43) evaluated for the parametric angle of $\theta = \pi/2$.

$$x_i(2) - x_i(1) = \frac{1}{2} \{u_i(1) + u_i(2) + [c(1) + c(2)]\beta_i\} [t(2) - t(1)] \quad (i=1,2,3) \quad (65)$$

For the first application of equation (65), the flow properties at point (1) are equated to those at point (2), whereas, for ensuing applications, the flow properties previously obtained at point (1) are employed. The flow properties at point (1) are obtained by interpolation using the quadratic bivariate polynomial given by equation (51). Since the axial step $[x(2) - x(1)]$ is determined by the CFL stability criterion, equation (65) is used to compute $[t(2) - t(1)]$, $y(1)$, and $z(1)$. The orientation of the parametric vector β_i in equation (65) is selected so that this vector lies in the meridional plane that contains the solution point. The unit vector α_i is obtained using the orthonormal relationship between α_i , β_i , and u_i/q ($i=1,2,3$).

At this stage, the wave surface compatibility equation corresponding to the parametric angle $\theta = \pi/2$ is applied between points (1) and (2). The appropriate equation is obtained by writing equation (54) in finite difference form and solving for the pressure at point (2). Denote this pressure by $P^*(2)$. The resulting equation contains cross derivatives (terms containing $\partial u_i / \partial x_j$) at both points (1) and (2). For the predictor, the cross derivatives at point (2) are equated to those at point (1), whereas, for the global corrector, the cross derivatives at point (2) are evaluated at that point by fitting interpolation polynomials in the solution plane.

The pressure $P(2)$ is calculated from the local Hugoniot equations. The pressure $P^*(2)$ is calculated from the wave surface compatibility

relation. The difference between $P(2)$ and $P^*(2)$ is driven to within a specified tolerance of zero using the secant method with iteration being performed on the shock wave angle $\phi(2)$. Two initial estimates of $\phi(2)$ are required to start the iterative process.

The shock wave point unit process is first applied as a predictor for each shock wave solution point. In this application, the value of α used in equation (59) is obtained by curve fitting points along space curve (A), and the cross derivatives at the solution point are equated to those at the bicharacteristic base point in the initial-value plane. After a tentative solution has been obtained at each shock wave point, a number of ensuing global corrections are performed. Here, the value of α used in equation (59) is based on data along space curve (B), and the cross derivative terms at the solution point are evaluated at that point. The resulting overall algorithm has second-order accuracy when the global correction is performed. The global iteration is terminated when successive values of α have converged at each of the shock wave solution points.

5. SOLID BODY-SHOCK WAVE POINT UNIT PROCESS

The solid body-shock wave point unit process is used to determine the flow properties downstream of the shock wave at a point where the shock wave intersects a solid boundary. This unit process is used to determine the solution for the points on the cowl on the downstream side of the cowl lip shock wave, and for the points on the centerbody or cowl on the downstream side of an internal reflected shock wave. The method of computation is essentially the same for either application. For the internal shock wave reflection, the flow properties

downstream of the incident shock wave, which constitute the upstream flow properties for the reflected shock wave, are computed by the modified field-shock wave point unit process discussed in Appendix E.

A depiction of the computational network used in the solid body-shock wave point unit process is presented in Figure 7. A typical solid body-shock wave solution point is denoted by point (P), with the outward unit normal vector to the solid boundary at this point denoted by $\hat{n}_b$. The locus of solid body-shock wave solution points represents the intersection of the shock wave with the solid boundary and defines space curve (A) in Figure 7. The intersection of the shock wave with the meridional plane passing through point (P) defines space curve (B). The unit vectors tangent to space curves (A) and (B) at point (P) are denoted by $\hat{l}$ and $\hat{t}$, respectively. The unit vector normal to the shock wave at point (P) is denoted by $\hat{n}_s$.

As for the bow shock wave point unit process, the unit vectors $\hat{l}$, $\hat{t}$, and $\hat{n}_s$ are referenced to the local coordinate system (x', y', z') , where x' , y' , and z' have the same definitions as noted before. Moreover, the tangential unit vector $\hat{t}$ again lies in the meridional plane and is defined by equation (58). In this scheme, however, the tangential unit vector $\hat{l}$ does not lie in the (y', z') -plane in most cases, but rather can have a nonzero x' -component. This tangential unit vector along space curve (A) may be represented by

$$\hat{l} = \frac{dx'}{ds} \hat{i}' + \frac{dy'}{ds} \hat{j}' + \frac{dz'}{ds} \hat{k}' \quad (66)$$

where ds is the differential arc length given by

$$(ds)^2 = (dx')^2 + (dy')^2 + (dz')^2 \quad (67)$$

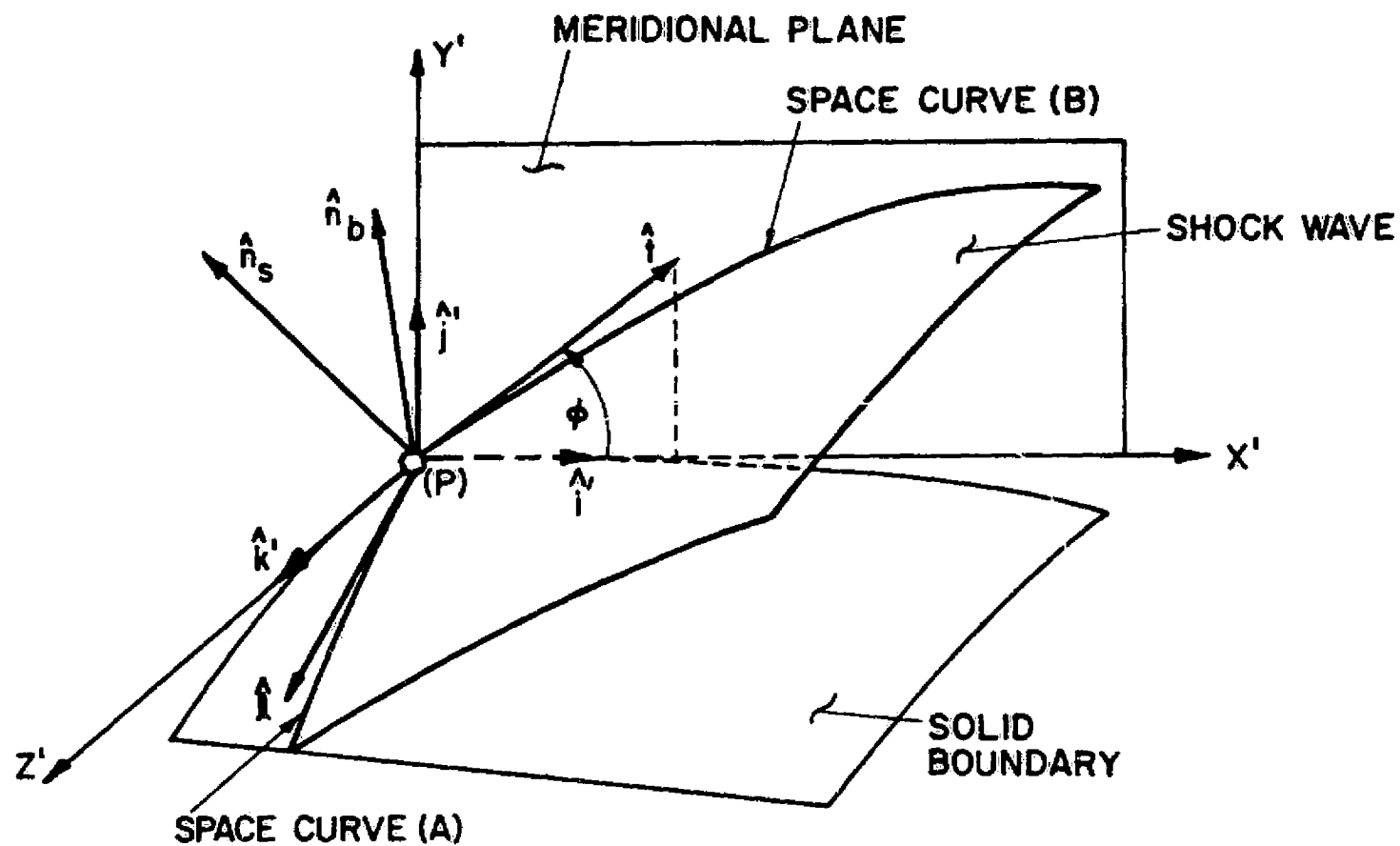


FIGURE 7. SOLID BODY-SHOCK WAVE POINT COMPUTATIONAL NETWORK

The derivatives in equation (66) are obtained by analytically differentiating the expressions

$$x'(\theta) = a_1 + a_2\theta + a_3\theta^2 \quad (68)$$

$$y'(\theta) = b_1 + b_2\theta + b_3\theta^2 \quad (69)$$

$$z'(\theta) = c_1 + c_2\theta + c_3\theta^2 \quad (70)$$

where coefficients a_i , b_i , and c_i ($i=1,2,3$) are obtained by curve fitting the respective expressions to three points on space curve (A). For the cowl lip shock wave points, space curve (A) is defined by the cowl lip itself, since the shock wave is assumed to be attached to the cowl lip. Alternatively, for computing the downstream flow properties at a reflected internal shock wave, space curve (A) is defined by the intersection of the incident shock wave with the solid boundary. The shock wave normal unit vector is found from equation (60).

The solid body-shock wave point unit process is initiated by determining the body normal unit vector $\hat{n}_b$ and the tangential unit vector $\hat{\ell}$. An assumption is then made for the shock wave angle ϕ in equation (58), and, by use of equation (60), the shock wave normal unit vector is determined. The local Hugoniot equations are then applied to obtain the downstream flow properties at point (P). The velocity normal to the wall is then obtained by forming the dot product of the body normal vector and the downstream velocity vector. The normal velocity is reduced to within a specified tolerance of zero by varying the shock angle ϕ using the secant iteration method.

6. INTERNAL FLOW SHOCK WAVE POINT UNIT PROCESSES

The unit process employed to compute the solution at a shock wave point in the internal flow field is similar to the bow shock wave point unit process. In the internal flow shock wave point unit process, however, the flow properties upstream of the shock wave at the solution point must be determined by the application of a modified interior point unit process. Moreover, modifications to the internal flow shock wave point unit process must be made when an internal flow shock wave solution point lies on or close to a solid boundary. The various versions of the internal flow shock wave point unit process are presented in Appendix E.

7. INTERNAL SHOCK MODIFIED-INTERIOR POINT AND -SOLID BODY POINT UNIT PROCESSES

In some situations during the computation of the internal flow field, the interior point and solid boundary point unit processes must be applied in a modified form. One such instance in which a modified form of the interior point unit process must be applied is shown in Figure 8. Here, the Mach cone, with apex at the interior solution point, intersects not only the initial-value plane but also the internal shock wave and a solid boundary. The unit process used in this case requires determining the bicharacteristic intersection points with the shock wave and the solid boundary in addition to the intersection points with the initial-value plane. Moreover, flow property values must be determined at all of these points. The bicharacteristic-shock wave and bicharacteristic-body intersection coordinates are calculated using the procedures discussed in Appendix D. The flow

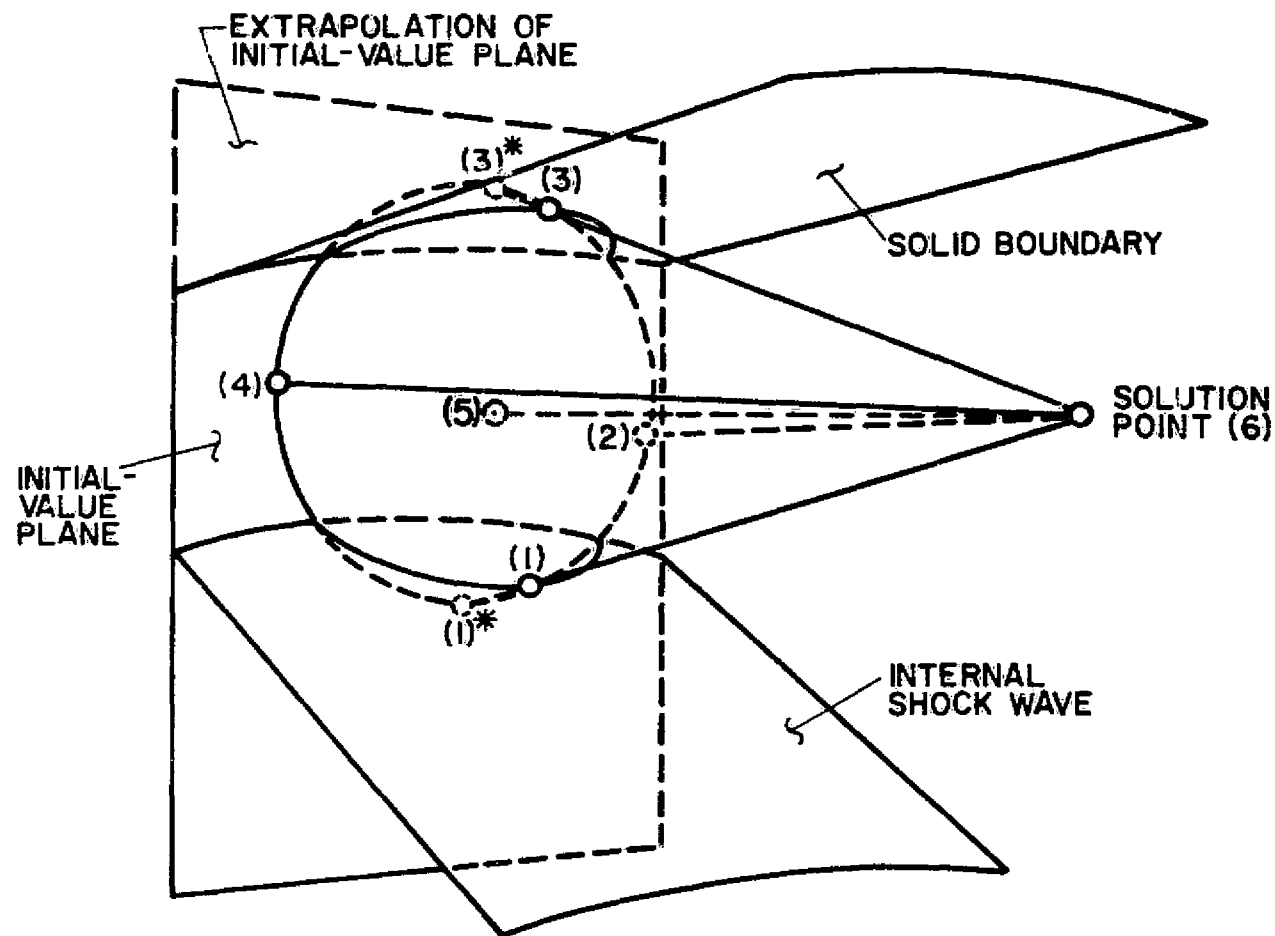


FIGURE 8. SHOCK-MODIFIED INTERIOR POINT COMPUTATIONAL NETWORK (STREAMLINE BASE POINT ON INITIAL-VALUE PLANE)

property values at these points are obtained by interpolation, either using a quadratic bivariate polynomial [equation (51)] for points on the initial-value plane, or using a quadratic trivariate polynomial for points on the shock wave surface or solid boundary surface. The various interpolation schemes are discussed in Appendix C. All of the unit processes, including the schemes incorporating the necessary modifications to handle the internal shock wave, are presented in Appendix E.

SECTION V

OVERALL NUMERICAL ALGORITHM

1. INTRODUCTION

The overall numerical algorithm consists of the repetitive application of the various unit processes to generate the global solution for given boundary conditions and a specified set of initial data.

The contours of the centerbody and the cowl, in addition to any planes of flow symmetry, constitute the boundaries of the computational flow regime. For the external flow field integration, the bow shock wave also represents a computational bound.

The initial data are specified on a plane of constant x . The x -coordinate axis is the longitudinal axis of the centerbody and the cowl (see Figure 1). Moreover, the mean flow direction is assumed to be in the x -coordinate direction.

An inverse marching scheme is employed in the numerical algorithm. The solution is obtained on space-like planes of constant x . The solution points on each plane represent the intersection points of continuous streamlines which are propagated from the data points specified on the initial-value plane. In addition to the streamline solution points, solution points are also obtained at the intersection of the external and internal shock waves with the solution plane, and for the internal flow field, on the space curves where the internal shock wave intersects the solid boundaries. These space curves are defined by the locus of

shock wave solution points.

Except in the vicinity of a shock wave reflection with a solid boundary, the axial (x) distance between the current initial-value plane and the current solution plane is determined by the application of the Courant-Friedrichs-Lewy (CFL) stability criterion (9). In the vicinity of a shock wave reflection with a solid boundary, the axial distance between successive solution planes is chosen so that the entire shock wave-solid boundary intersection falls between two adjacent solution planes.

The external flow about the forebody is computed first. The external flow field integration requires the periodic addition of streamlines in order to retain a well dispersed computational mesh. Furthermore, periodic deletion of selected streamlines is also required so that the number of computational points lies within bounds.

The internal flow field can be computed with or without the discrete fitting of the internal shock wave system. The option in which shock waves are not discretely fitted may be used in cases in which the internal shock waves are quite weak in strength, and thereby an acceptable solution can be obtained by smearing the internal discontinuities.

In this section, brief discussions are presented on generation of the initial data, boundary conditions, regulation of the marching step size, computation of the transport forcing functions, and numerical stability. A detailed discussion of the overall numerical algorithm is presented in Apperdx F.

2. INITIAL-VALUE PLANE

The initial data are specified on a plane of constant x (see Figure 1). The flow must be supersonic at every point on this plane. For uniqueness and existence of a genuine solution, the values of the five dependent variables (u , v , w , P , and ρ) prescribed on this surface must have at least continuous first partial derivatives.

If the forebody flow field is to be computed, the initial-value plane must be specified at an axial (x) station that is upstream of the forebody flow computational regime (see Figure 1). The last solution plane of the forebody flow field computation is adjusted to lie at the axial station of the cowl lip, and constitutes the initial-value plane for the internal flow field computation. The cowl lip is assumed to be contained in a plane of constant x . Furthermore, the bow shock wave must fall outside of the cowl lip, or, in the limit, intersect the cowl lip exactly. The internal flow cannot be calculated if the bow shock wave is ingested into the annulus. The points on the solution plane at the cowl lip axial station are redistributed to obtain a ring of solution points coincident with the cowl lip.

If the forebody is conical ahead of the axial station where the initial-value plane is specified, an approximate flow property field on this plane may be internally generated in the computer program. The internally generated initial data are obtained by an approximate technique which employs the Taylor-Maccoll solution for the flow about a circular cone at zero incidence. A superposition method is then used to obtain an approximation for the flow about a circular cone at nonzero angle of attack by neglecting the cross flow effects. Alternatively, a more exact solution for the initial data for flow about a

circular cone at incidence may be obtained by employing the results of Jones (33).

If the forebody is not conical ahead of the axial station of the initial-value plane, another source of initial data must be used. If available, experimental data may be employed.

3. SOLID BOUNDARY SURFACES

The computer program developed in the present investigation assumes that both the centerbody and the cowl are axisymmetric. For the purposes of geometry description, the axial (x) domain is divided into a number of intervals. In any interval, the body radius r may be specified by either tabular input, or by supplying the coefficients in the cubic polynomial

$$r(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad (71)$$

where the subscript i denotes the i th interval, $r(x)$ is the body radius at axial position x ($x_i \leq x < x_{i+1}$), and the coefficients a_i , b_i , c_i , and d_i are obtained by curve fitting the body contour. Since equation (71) is a cubic, slope and curvature can be matched at the junction point between two adjacent intervals (i.e., spline fits can be employed).

4. INTEGRATION STEP SIZE REGULATION

Except in the vicinity of a reflection of the internal shock wave with a solid boundary, the axial marching step between successive solution planes is determined by the application of the Courant-Friedrichs-Lewy (CFL) stability criterion (9). The CFL stability criterion mandates that the domain of dependence of the differential equations be contained within the convex hull of the finite difference

network. That is, the Mach cone must be inside the outer periphery of the nine initial-value plane field points used in formulating the bi-variate interpolation polynomial, equation (51). The allowable axial step is given by

$$\Delta x = [u^2/(cq)][1 - (c/q)(q^2/u^2 - 1)^{1/2}]R_{\min} \quad (72)$$

where Δx is the marching step, and $R_{\min}$ is the distance between the streamline intersection point with the initial-value plane and the nearest point on the convex hull of the finite difference network. Equation (72) is applied at every streamline point on the initial-value plane, with the actual integration step being chosen as the Δx value at the most restrictive point. Equation (72) is applied only to streamline points. The shock wave points are excluded. Moreover, in the internal flow field integration, the shock wave points are ignored in defining the convex hull of the finite difference network when applying the stability criterion to a streamline point.

In the vicinity of a reflection of the internal shock wave with a solid boundary, the axial step is controlled by the constraint that the shock wave-solid body intersection is contained entirely between two adjacent solution planes. The fit point stencils used in formulating the various interpolation polynomials are appropriately expanded, in this case, so that the CFL stability criterion is satisfied.

5. CALCULATION OF THE TRANSPORT FORCING FUNCTIONS

The numerical procedure developed in the present investigation has the capability to include the influence of molecular transport on the solution by treating the viscous and thermal diffusion terms in the governing partial differential equations as forcing functions, or

correction terms, in the method of characteristics scheme. The computer program has the capability to include the influence of viscous and thermal diffusion in the computation of the external flow about the forebody, and in the computation of the internal flow field in which shock waves are not discretely fitted. The program option in which shock waves are discretely fitted in the internal flow field does not have the capability to include the influence of molecular transport in the computation, but rather assumes the flow to be inviscid and adiabatic. The detailed calculation procedures used for obtaining the transport forcing terms are presented in Appendix G.

6. NUMERICAL STABILITY

A stability analysis of the nonlinear finite difference algorithm including molecular transport was not attempted. Instead, a stability analysis for isentropic flow was conducted. Stability of the generalized analysis was then verified by actual numerical calculations.

Ransom, Hoffman, and Thompson (9) used the present numerical method to compute the continuous steady three-dimensional supersonic isentropic flow in a nozzle. The CFL stability criterion was used for locating successive solution planes. A von Neumann linear stability analysis indicated that interpolated flow properties, instead of the actual known values, should be used at the streamline-initial-value plane intersection point [point (5) in Figure 3]. The present analysis uses interpolated flow properties at all points in the initial-value plane.

SECTION VI

COMPUTATIONAL RESULTS

1. INTRODUCTION

Selected computational results are presented and discussed in this section. The results presented are divided into three major categories: external flow about the forebody, continuous internal flow, and internal flow in which the internal shock wave system has been computed. In some instances, both axisymmetric flow and three-dimensional flow results are shown. For the internal flow field in which shock waves have been fitted, some comparisons with experimental data are made. Moreover, some results of the present analysis are compared with those of existing computational methods.

2. EXTERNAL FLOW ABOUT THE FOREBODY

For the purpose of testing the external flow integration procedure, the flow field about a right circular cone at incidence was computed. This is a conical flow field and the solution is constant along rays emanating from the vertex of the cone (i.e., there is no characteristic length, so the solution has no dependency on x). At zero angle of attack, the solution depends only on the angle subtended by a given ray and the x -axis. At nonzero incidence, an azimuthal variation also exists. To obtain the required initial data, the results of Jones (33) were employed. The computed results should maintain the conical nature of the flow field.

Figure 9 presents numerical results obtained for a 10.0° half-angle cone at 2.5° angle of attack α with a free-stream Mach number M_∞ of 3.0. The computation employed 21 circumferential stations in the computed sector (half-plane), and the number of radial stations on the initial-value plane was 11. The computed static pressure P normalized by the free-stream static pressure P_∞ is plotted versus the axial position x normalized by the cowl lip radius R_c . The pressure distributions on the rays formed by the forebody and the bow shock wave on both the leeward and windward planes of symmetry are shown. Since the flow is conical, the solution should remain constant along each of these four rays at the respective pressure values at the appropriate points on the initial-value plane. The initial-value plane pressures are denoted by the straight line segments. The method of characteristics solution is shown at a discrete number of axial stations, each station corresponding to the axial location of a given solution plane. The method of characteristics solution maintains the conical nature of the flow field.

It should be noted that the increase in pressure across the leeward side of the bow shock wave is minimal. As the angle of incidence is further increased, the strength of the bow shock wave on the leeward side is reduced until the point is reached where the angle of attack is equal to the cone half-angle. At this point, no shock wave exists on the leeward meridional plane. Further increase in the angle of incidence causes a flow expansion to occur on the leeward side. Since the present analysis assumes that a shock wave exists about the

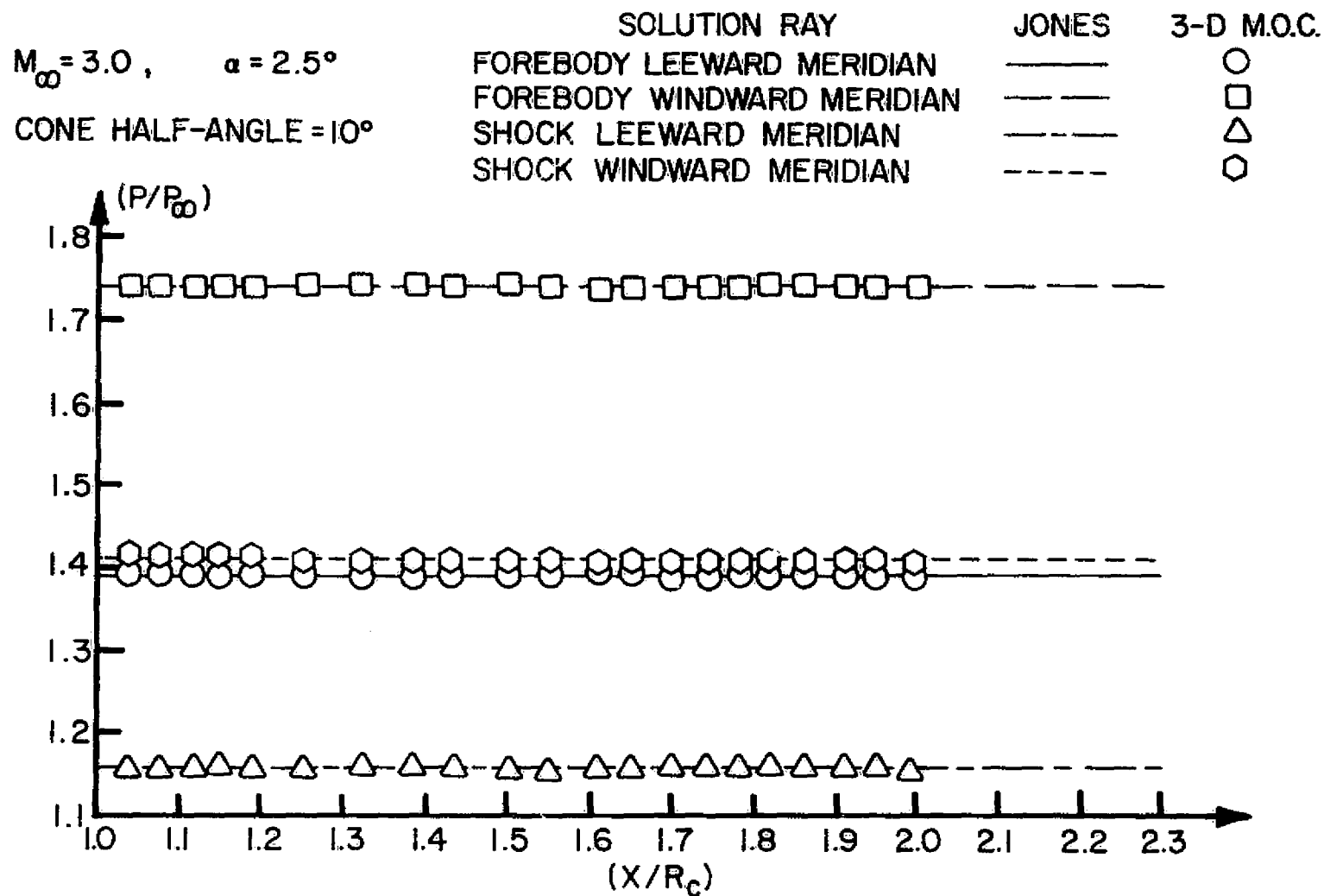


FIGURE 9. PRESSURE DISTRIBUTIONS FOR EXTERNAL FLOW

entire forebody, the case where a flow expansion occurs on the leeward side cannot be computed.

The external flow about a circular cone at incidence was also computed including the effects of molecular diffusion. No significant changes in the computed results were noted. Approximately 60 percent more computer execution time was required for the computation which included the molecular diffusion terms.

3. CONTINUOUS INTERNAL FLOW

For the purpose of testing the continuous internal flow integration procedure in which shock waves are not discretely fitted, the axisymmetric flow field in the simplified geometry inlet illustrated in Figure 10 was computed. The geometry of this inlet was selected so that the slope of the cowl contour at the cowl lip was equal to the slope of the streamline at the cowl lip. Hence, no flow turning occurs at the cowl lip and the internal shock wave system is not generated.

Figure 10 illustrates the inlet geometry and the pressure distributions on the centerbody and the cowl. A monotonic increase in pressure on the surface of the cowl occurs. The pressure on the centerbody retains its conical flow value until the Mach wave emanating from the cowl lip reaches the centerbody. After that point, a monotonic increase in the centerbody pressure occurs. This computation was performed with 21 radial stations and 1 circumferential station. The maximum deviation in mass flow rate on any solution plane from the mass flow rate across the cowl lip solution plane was 0.25 percent.

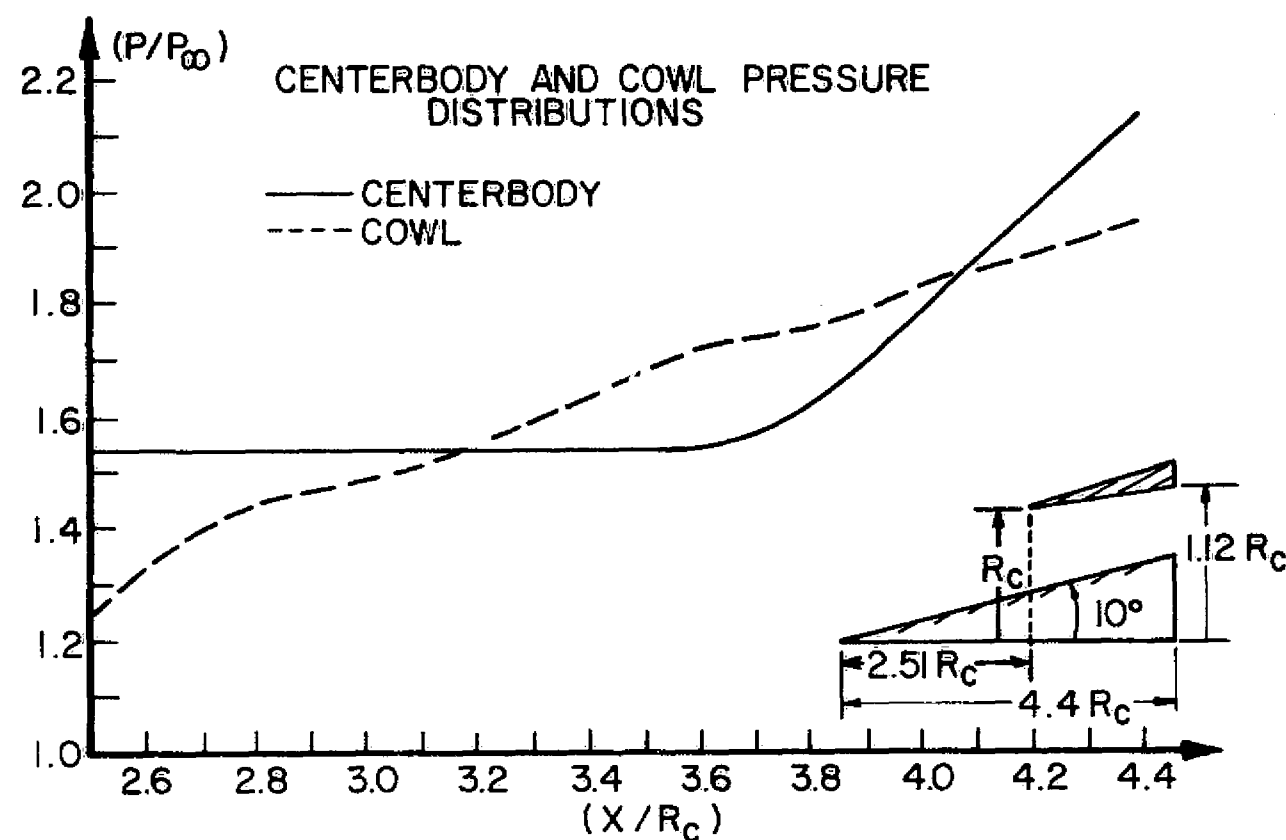


FIGURE 10. AXISYMMETRIC CONTINUOUS INTERNAL FLOW PRESSURE DISTRIBUTIONS FOR $M_\infty=3.0$ AND $\alpha=0^\circ$

The flow field in the simplified geometry inlet illustrated in Figure 10 was also computed including the effects of molecular diffusion on the solution. No significant changes in the computed results were noted. The increase in computer execution time was approximately 60 percent.

4. INTERNAL FLOW WITH DISCRETE FITTING OF THE INTERNAL SHOCK WAVE SYSTEM

Internal flow calculations were performed for the Boeing Mach 3.5 supersonic mixed-compression inlet documented in Reference (34). The centerbody and cowl coordinates of this inlet are listed in Table 1. The boundary contours are illustrated in Figure 11 for the design case of zero centerbody translation. This inlet has a forebody which is conical (the forebody is not shown in Figure 11). Consequently, all of the numerical solutions were started at the cowl lip axial station. The initial data were obtained by employing the results of Jones (33).

Extensive boundary layer removal is employed in this inlet to control boundary layer separation in regions of strong adverse pressure gradients such as those caused by oblique shock wave-boundary layer interactions. Figure 11 indicates regions where the boundary layer is removed. Since the present analysis does not compute the boundary layer nor takes account of boundary layer removal, good agreement between computed and experimental results cannot be expected in regions of high viscous interaction. For this inlet, 13.3 percent of the cowl lip mass flow rate was removed through boundary layer bleed at the design point condition to control boundary layer separation (34).

TABLE 1

MACH 3.5 INLET COORDINATES

CENTERBODY		CENTERBODY		COWL		COWL	
x/R_c	r/R_c	x/R_c	r/R_c	x/R_c	r/R_c	x/R_c	r/R_c
0.0	0.0	4.8	0.7504	2.86	1.0	4.55	0.8695
4.0	0.70532	4.9	0.7391	3.1	1.004188	4.6	0.864
4.1	0.7228	5.1	0.7120	3.2	1.0054	4.65	0.86
4.2	0.7387	5.3	0.6829	3.4	1.0051	4.7	0.8572
4.3	0.7512	5.5	0.6525	3.6	0.99996	4.8	0.8533
4.4	0.759	5.6	0.6362	3.8	0.9882	4.9	0.8511
4.5	0.7625	5.7	0.618	4.0	0.9681	5.0	0.8502
4.55	0.763	5.8	0.5973	4.1	0.954	5.1	0.85
4.6	0.7625	5.9	0.5744	4.2	0.9364	5.6	0.85
4.65	0.7611	6.0	0.5467	4.25	0.9261	5.8	0.8574
4.7	0.7585			4.3	0.9154	5.9	0.8646
				4.4	0.8949	6.0	0.8735
				4.5	0.8768		

x : Axial Position

r : Radial Position

R_c : Radius of Cowl Lip

The first results employing the internal flow computational algorithm in which shock waves are discretely fitted are for the design conditions of $M_\infty = 3.5$, zero centerbody translation, and zero incidence ($\alpha = 0^\circ$). At the design point, the bow shock wave intersects the cowl lip exactly at zero incidence. Since the flow field is axisymmetric at zero incidence, it can be computed using a two-dimensional method. Comparisons of the results obtained from the present analysis with those obtained from a two-dimensional method of characteristics scheme (35) for the zero incidence design point conditions are shown in Figures 12 and 13. In these figures, the static pressure P normalized by the free-stream stagnation pressure P_{T_∞} is plotted versus the axial position x normalized by the cowl lip radius R_c . Pressure distributions are shown for both the centerbody and the cowl. The results obtained

CENTERBODY TRANSLATION = 0
 (*) - BOUNDARY LAYER BLEED REGION

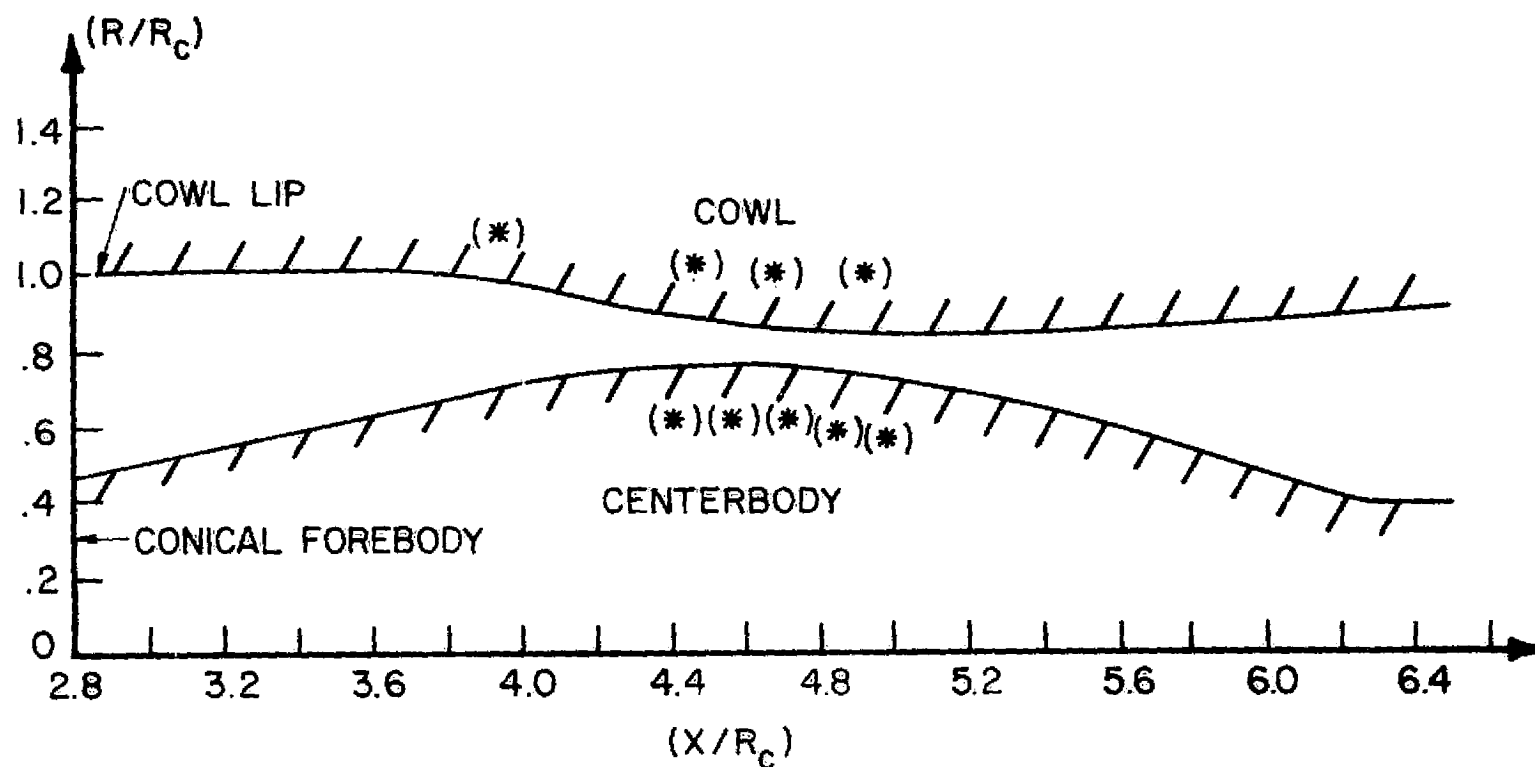


FIGURE II. BOEING MACH 3.5 MIXED-COMPRESSION INLET
 CENTERBODY AND COWL CONTOURS

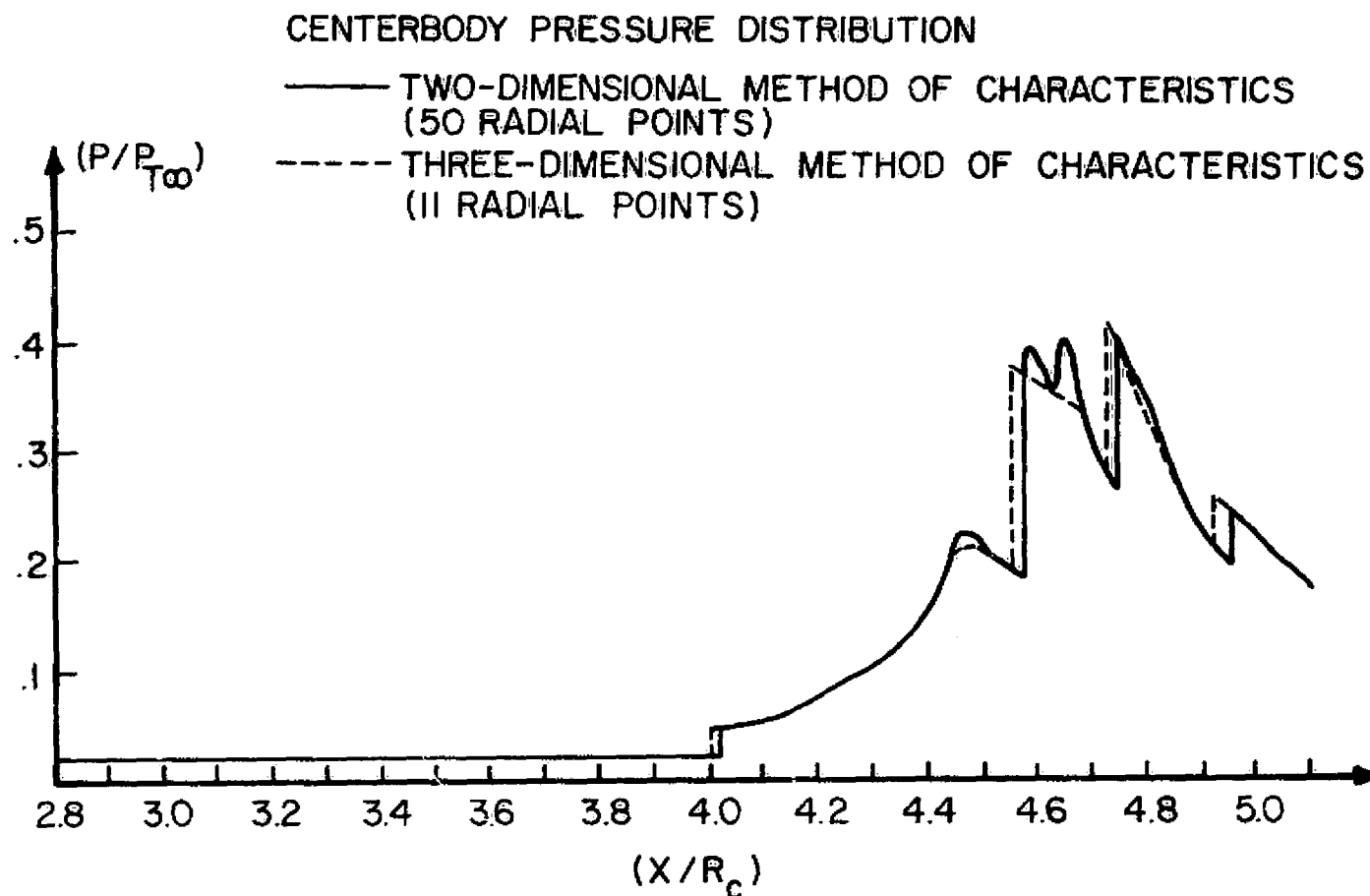


FIGURE 12. COMPARISON OF THE TWO- AND THREE-DIMENSIONAL METHODS OF CHARACTERISTICS FOR $M_\infty = 3.5$, $\alpha = 0^\circ$, TRANSLATION = 0

COWL PRESSURE DISTRIBUTION

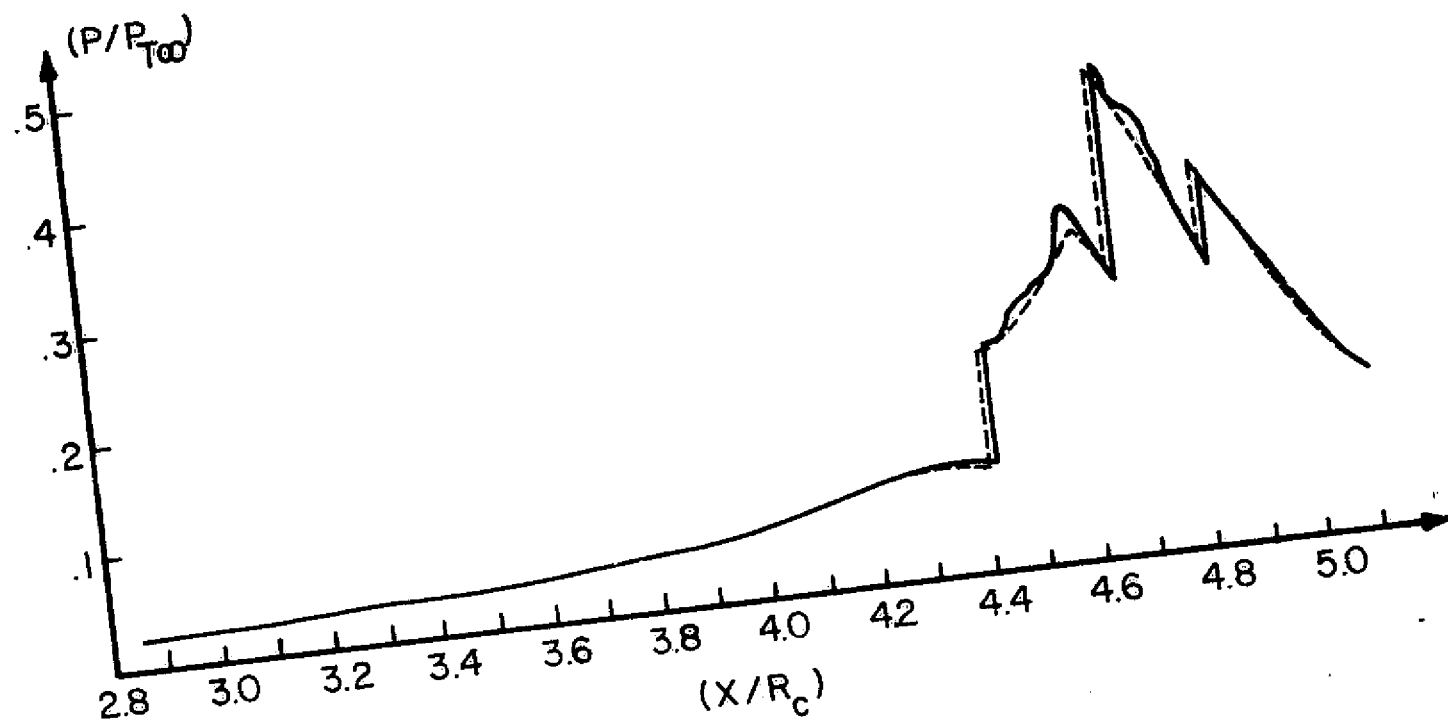


FIGURE 12. (CONTINUED)

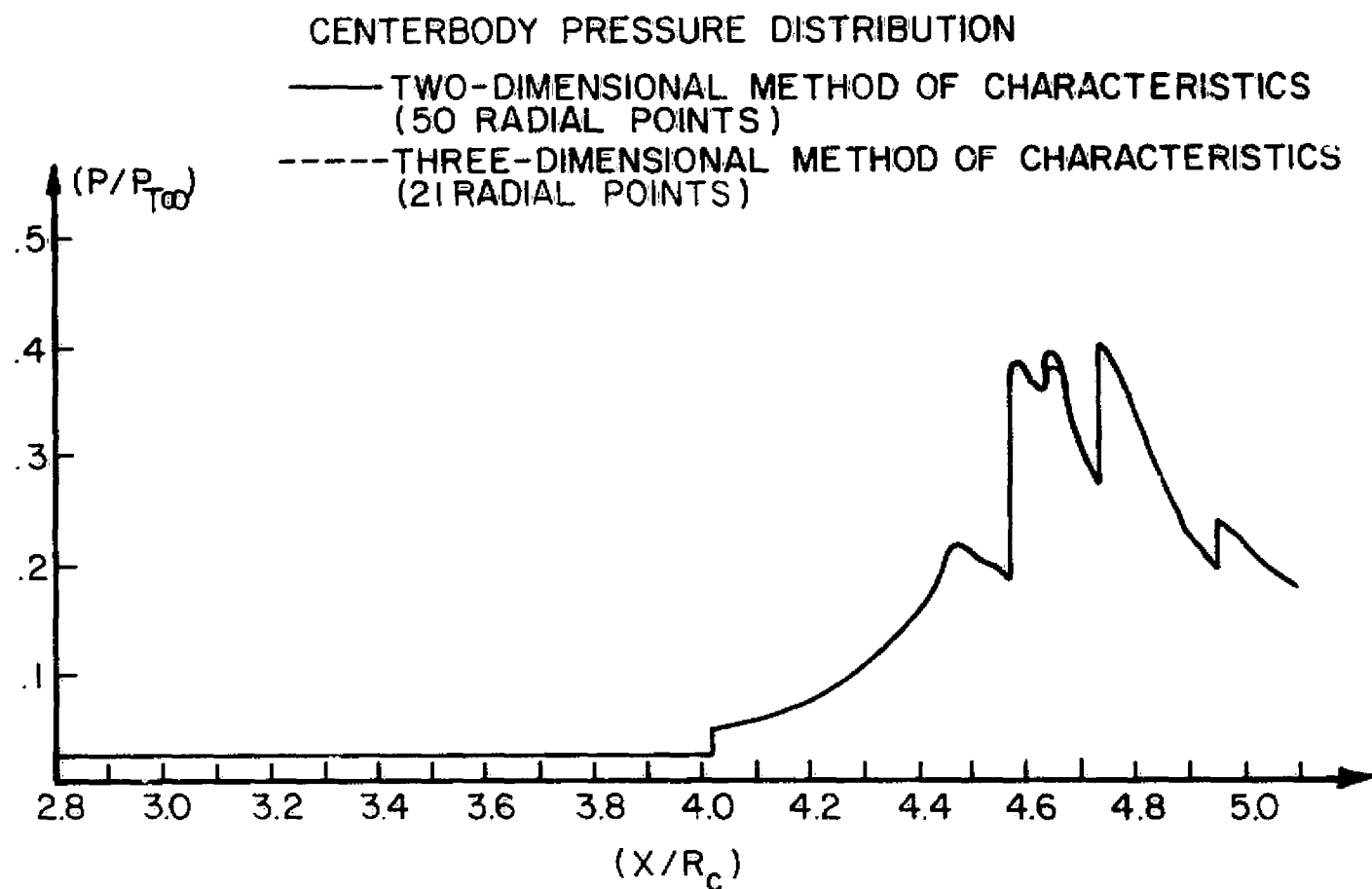


FIGURE 13. COMPARISON OF THE TWO- AND THREE-DIMENSIONAL METHODS OF CHARACTERISTICS FOR $M_{\infty} = 3.5$, $\alpha = 0^\circ$, TRANSLATION = 0

COWL PRESSURE DISTRIBUTION

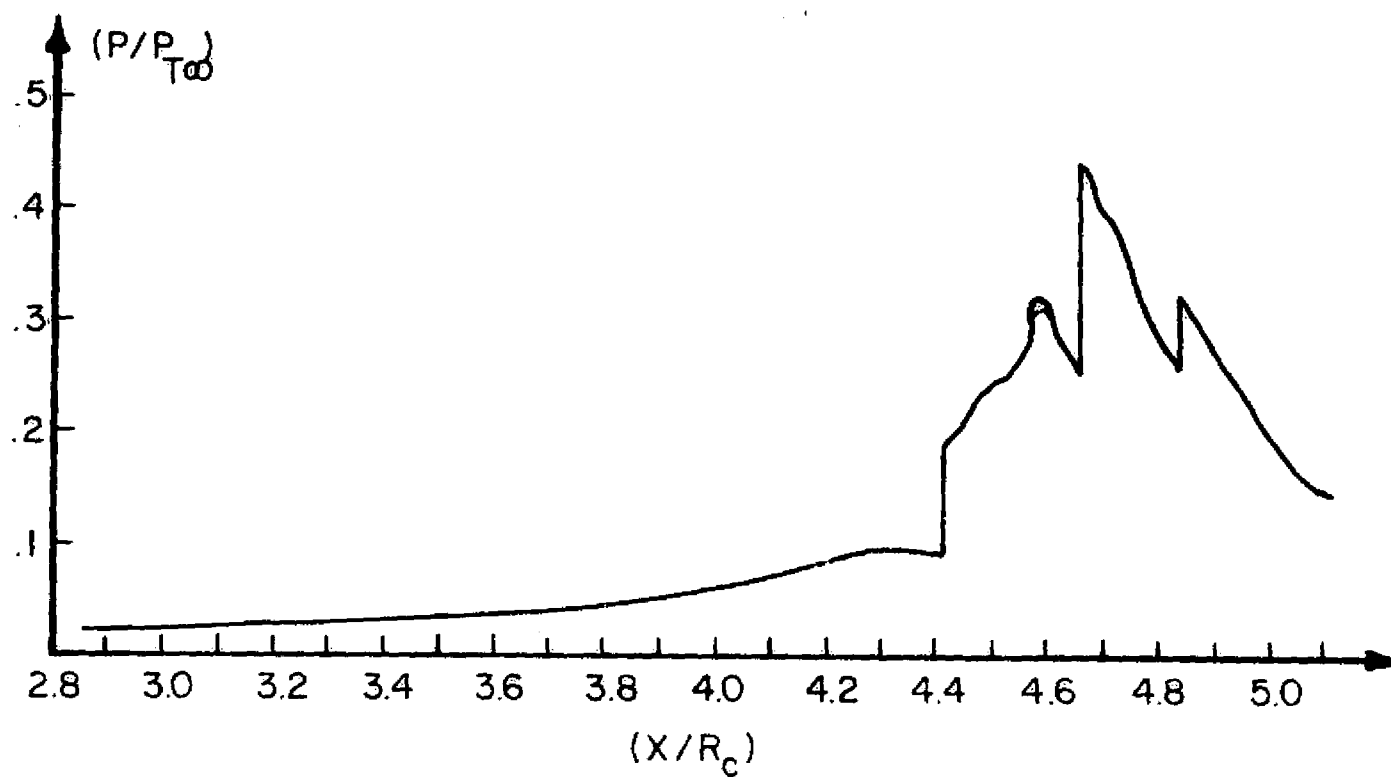


FIGURE 13. (CONTINUED)

by the two-dimensional method of characteristics algorithm are indicated by solid lines, and the results obtained by the present analysis are indicated by the dashed lines. Fifty radial stations were used in the two-dimensional method of characteristics solution. Figure 12 illustrates the case where a total of 11 radial stations (9 streamline points and an upstream and downstream shock wave point) were employed in the three-dimensional method of characteristics solution. Good overall agreement is observed. A slight smearing of the pressure distribution downstream of the second intersection of the shock wave with the centerbody and a slight shifting of the shock wave-solid body intersections are present in the three-dimensional algorithm's results. The smearing of the pressure distribution is primarily a consequence of the coarse mesh size used in the three-dimensional scheme's solution. Figure 13 illustrates the solution obtained by the three-dimensional analysis when a total of 21 radial stations were used in the computation. In this case, the agreement between the three-dimensional analysis and the two-dimensional analysis is excellent. The pressure distribution behind the second shock wave-centerbody intersection is predicted very well. The axial locations of the shock wave-solid boundary intersections also agree very well. For this computation, the maximum deviation in the computed mass flow rate at any solution plane from that at the cowl lip solution plane was approximately 0.77 percent.

Comparisons of the results of the present analysis with experimental data (36) for the Boeing Mach 3.5 inlet for $\alpha = 0^\circ$ are shown in Figure 14. Generally speaking, good agreement is observed. The three-dimensional method of characteristics scheme predicts shock wave-solid

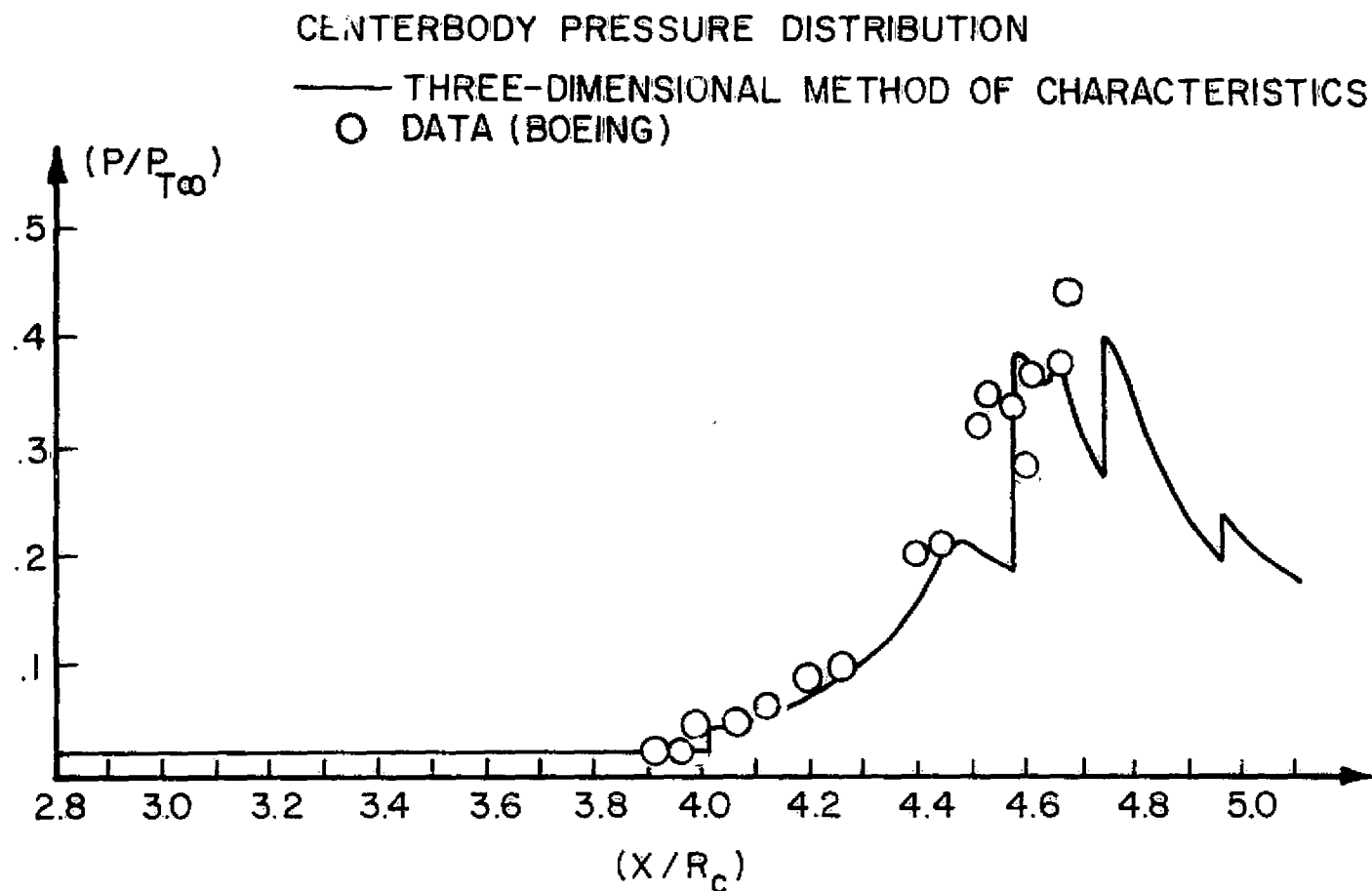


FIGURE 14. COMPARISON OF THE THREE-DIMENSIONAL METHOD OF CHARACTERISTICS WITH EXPERIMENTAL DATA FOR $M_\infty = 3.5$, $\alpha = 0^\circ$, TRANSLATION = 0

COWL PRESSURE DISTRIBUTION

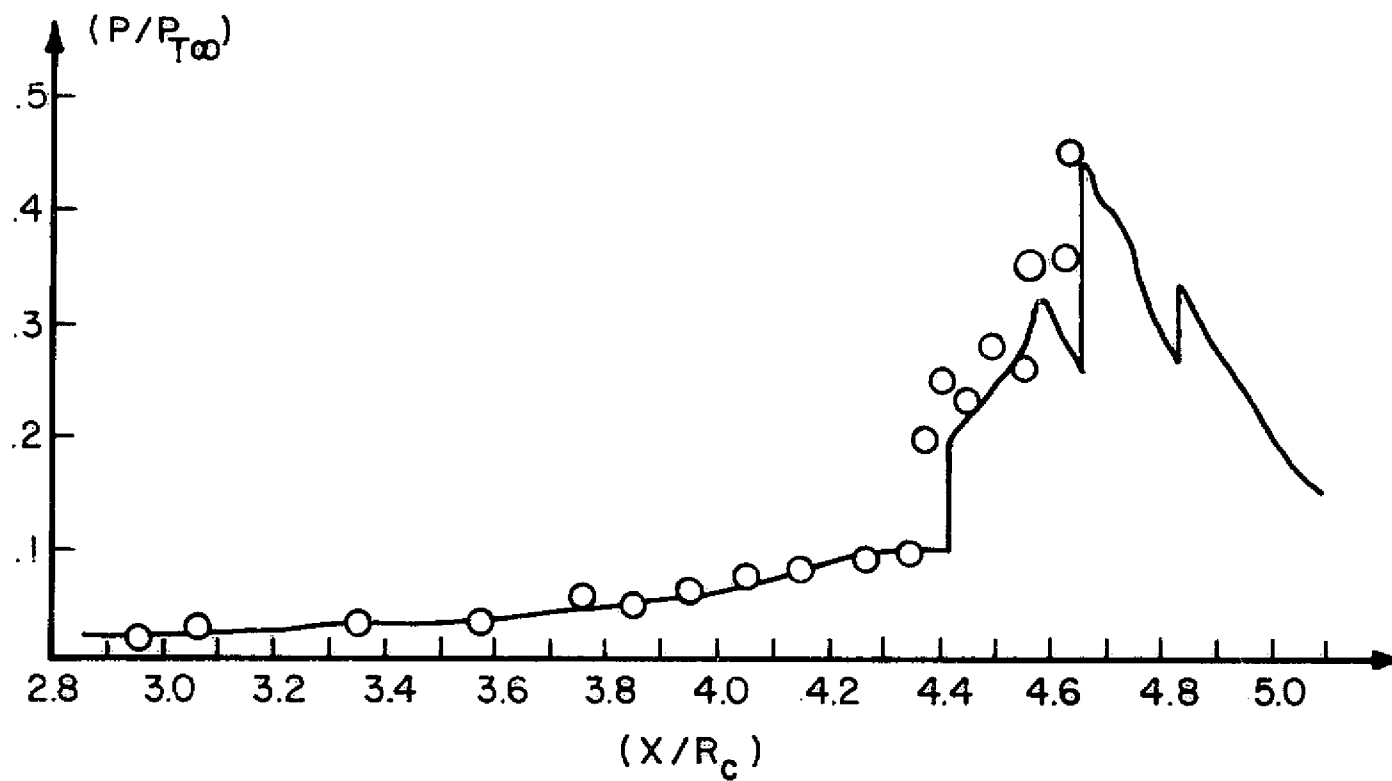


FIGURE 14. (CONTINUED)

boundary intersections slightly downstream of the locations where the experimental data indicate the intersections to occur. Since the presence of a boundary layer would move the predicted intersection points forward, this result seems plausible. Note that the best agreement is obtained away from the regions where the boundary layer is removed (see Figure 11).

At a given free-stream Mach number, the centerbody assembly must be translated forward of its design point position as the angle of incidence is increased to maintain supersonic flow through the geometric throat of the annulus. The forward translation of the centerbody causes the cross-sectional area of the geometric throat to increase. Moreover, as the free-stream Mach number is reduced from the design point value, even further forward translation of the centerbody is required. An experimentally obtained centerbody translation schedule (37) is presented in Figure 15, where the nondimensional centerbody translation is denoted by $\Delta x/R_c$. The effects of an increase in the angle of incidence and a reduction of the free-stream Mach number are clearly illustrated in this figure.

Results are presented below for two off-design conditions:

- (1) $M_\infty = 2.5$ with a centerbody translation of $\Delta x/R_c = 0.855$, and
- (2) $M_\infty = 3.3$ with a centerbody translation of $\Delta x/R_c = 0.356$. For each of these off-design conditions, the internal flow field is computed for incidence angles of $\alpha = 0^\circ$, 3.0° , and 5.0° . For both off-design conditions, the results of the present analysis are compared with experimental data for an incidence angle of $\alpha = 3.0^\circ$.

Results for the first off-design case of $M_\infty = 2.5$ and $\Delta x/R_c = 0.855$ are presented in Figures 16 to 19. Figure 16 illustrates the

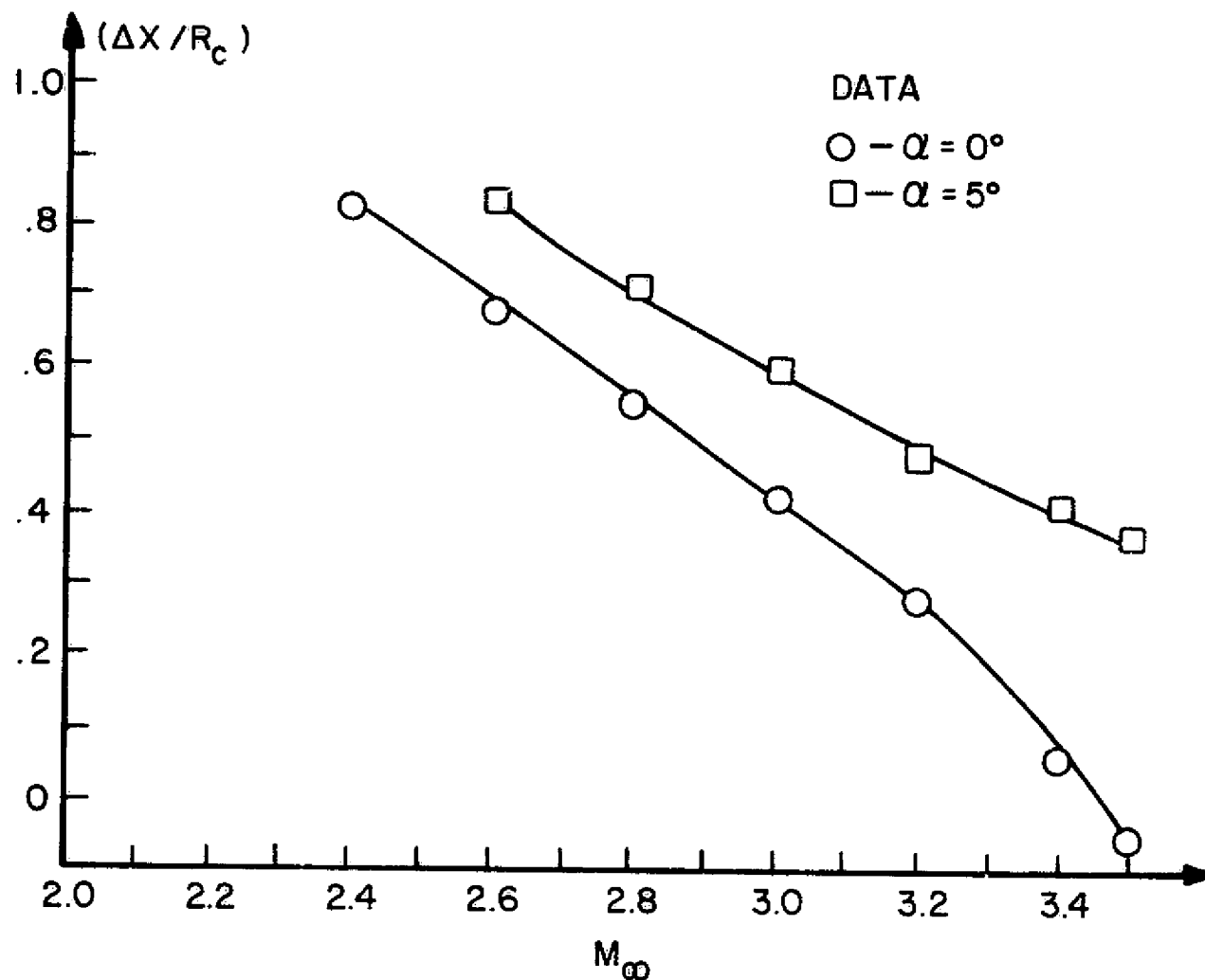


FIGURE 15. EXPERIMENTAL TRANSLATION SCHEDULE
FOR MACH 3.5 INLET

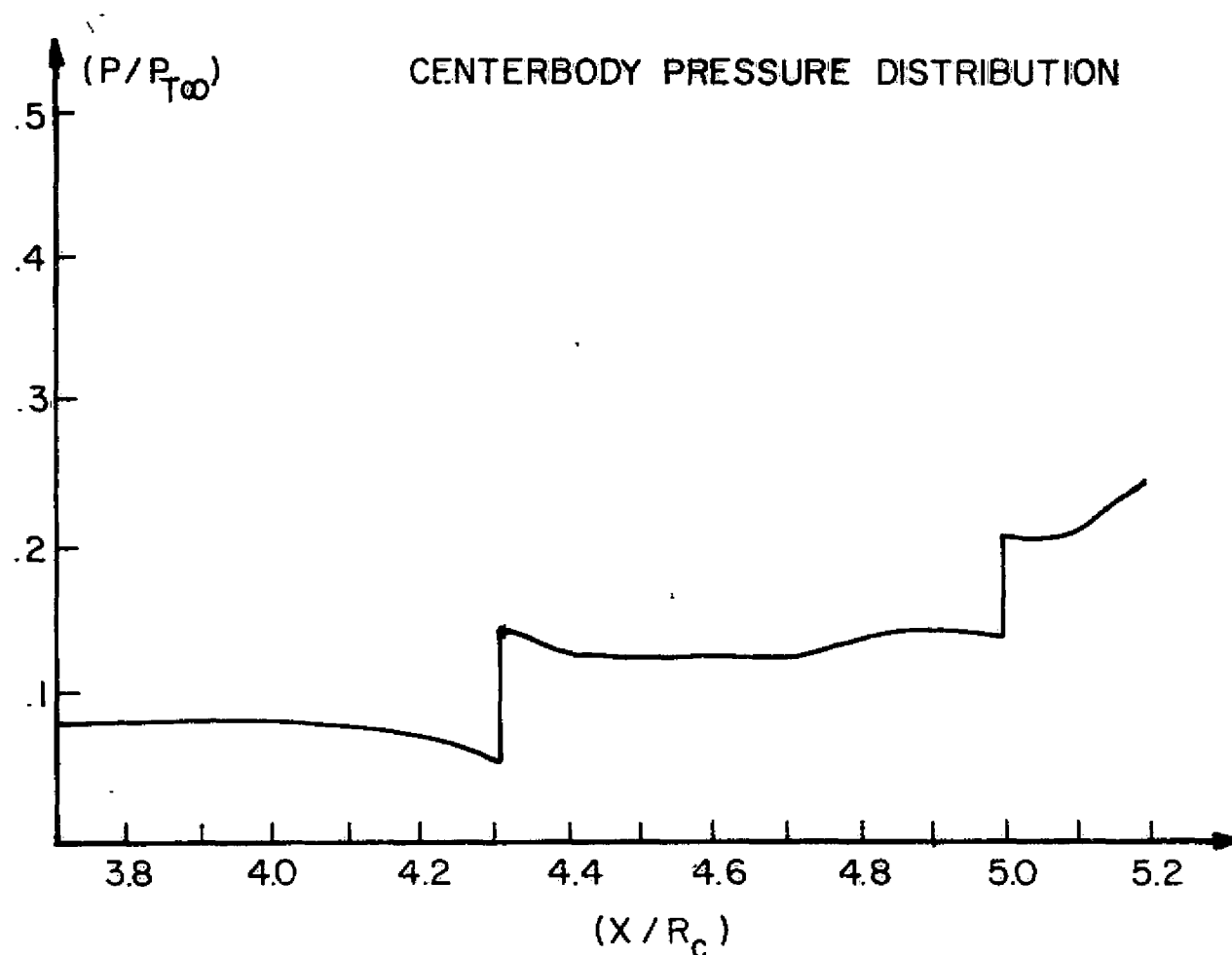


FIGURE 16. THREE-DIMENSIONAL METHOD OF CHARACTERISTICS
SOLUTION FOR $M_\infty = 2.5$, $\alpha = 0^\circ$, $\Delta X/R_c = .855$

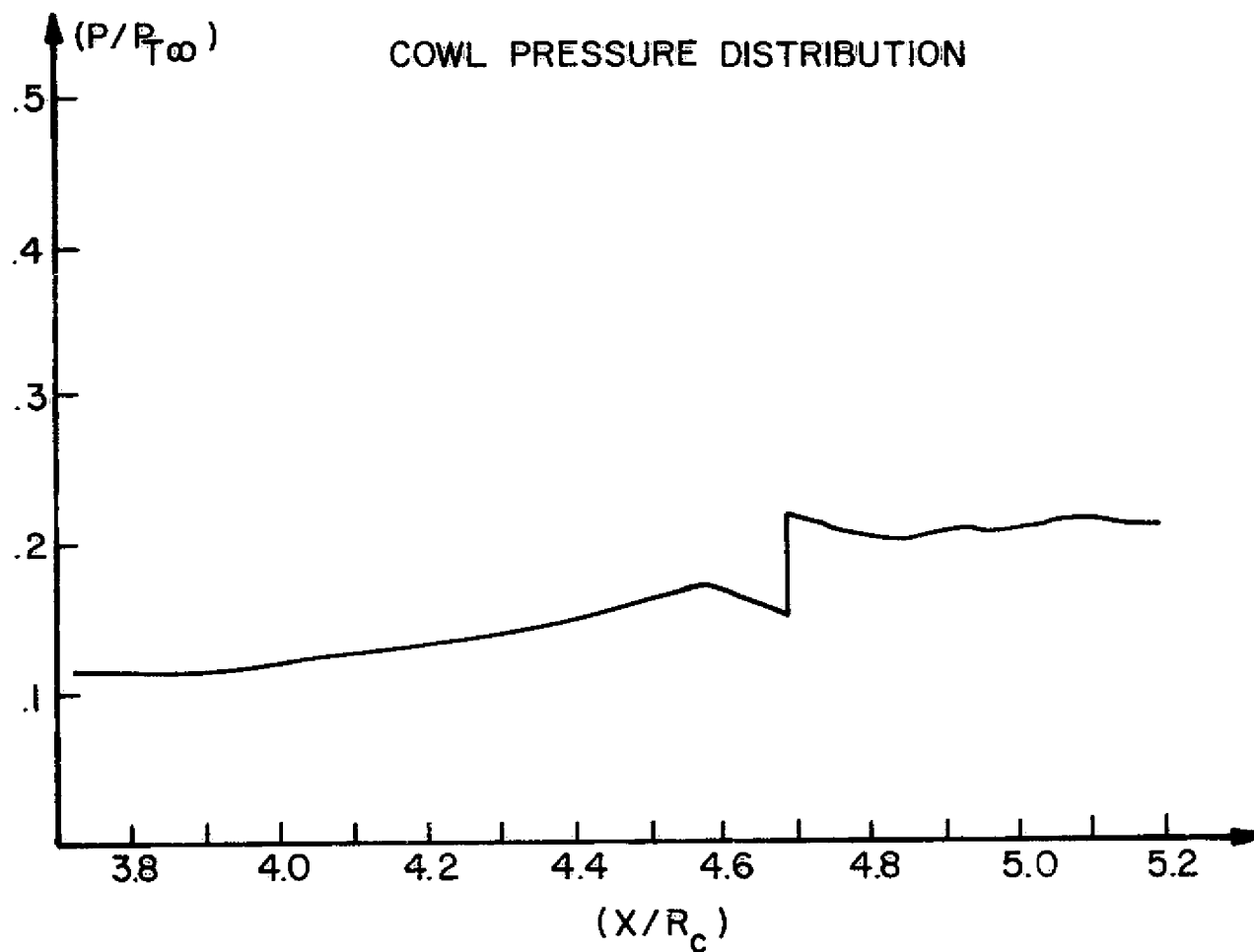


FIGURE 16. (CONTINUED)

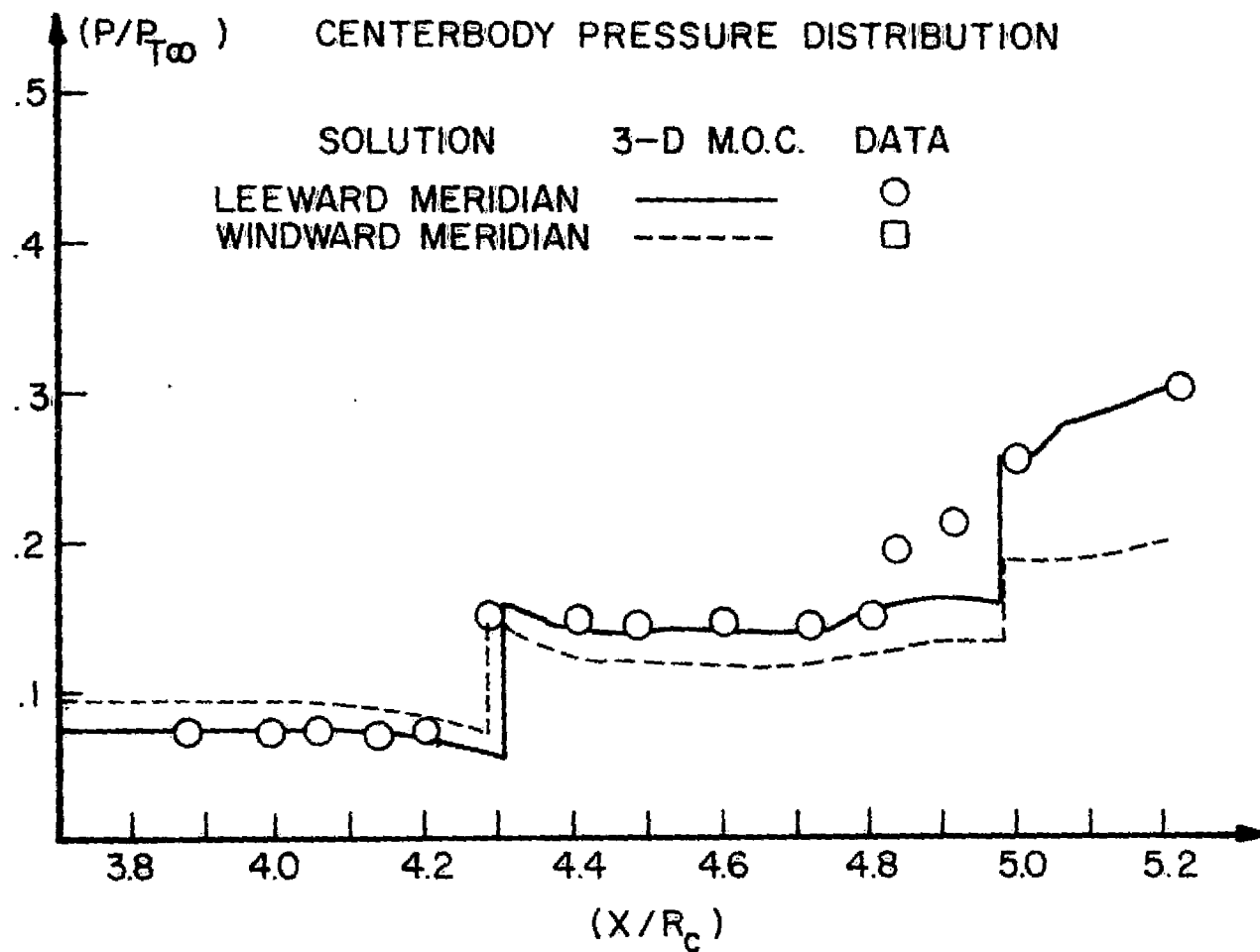


FIGURE 17. COMPARISON OF THREE-DIMENSIONAL METHOD OF CHARACTERISTICS SOLUTION WITH EXPERIMENTAL DATA FOR $M_\infty = 2.5$, $\alpha = 3^\circ$, $\Delta X/R_c = .855$

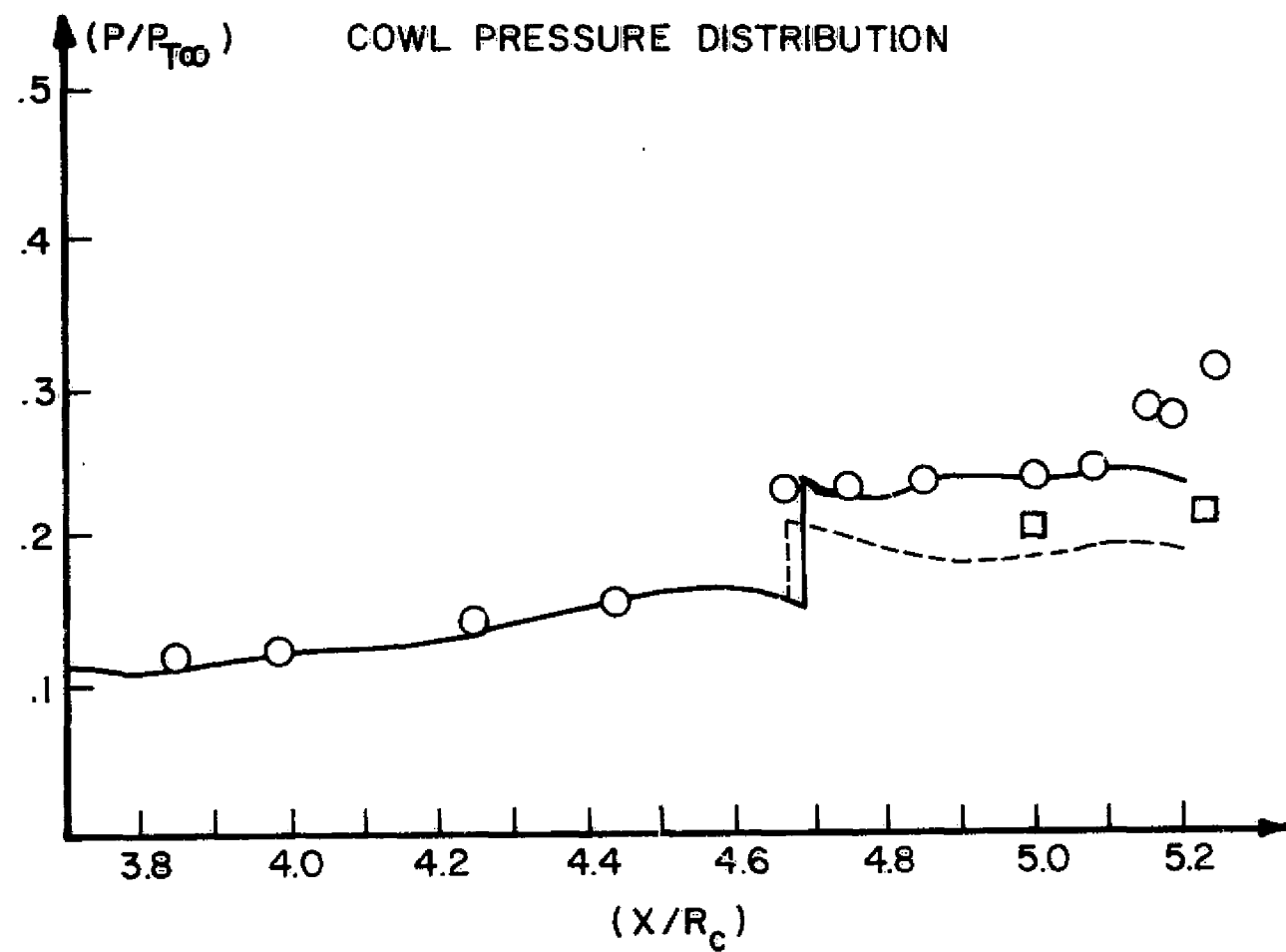


FIGURE 17. (CONTINUED)

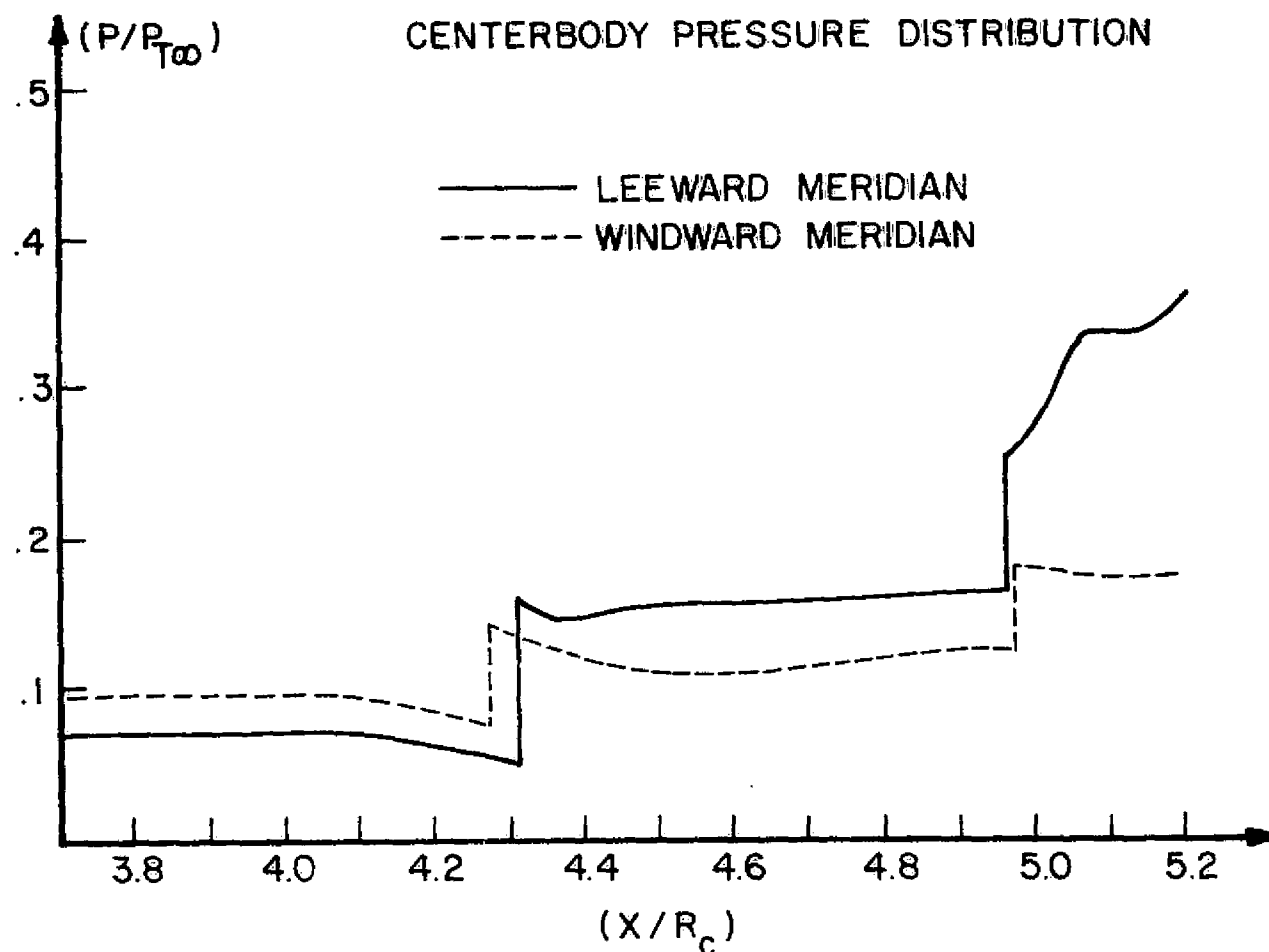


FIGURE 18. THREE-DIMENSIONAL METHOD OF CHARACTERISTICS
SOLUTION FOR $M_\infty = 2.5$, $\alpha = 5^\circ$, $\Delta X/R_c = .855$

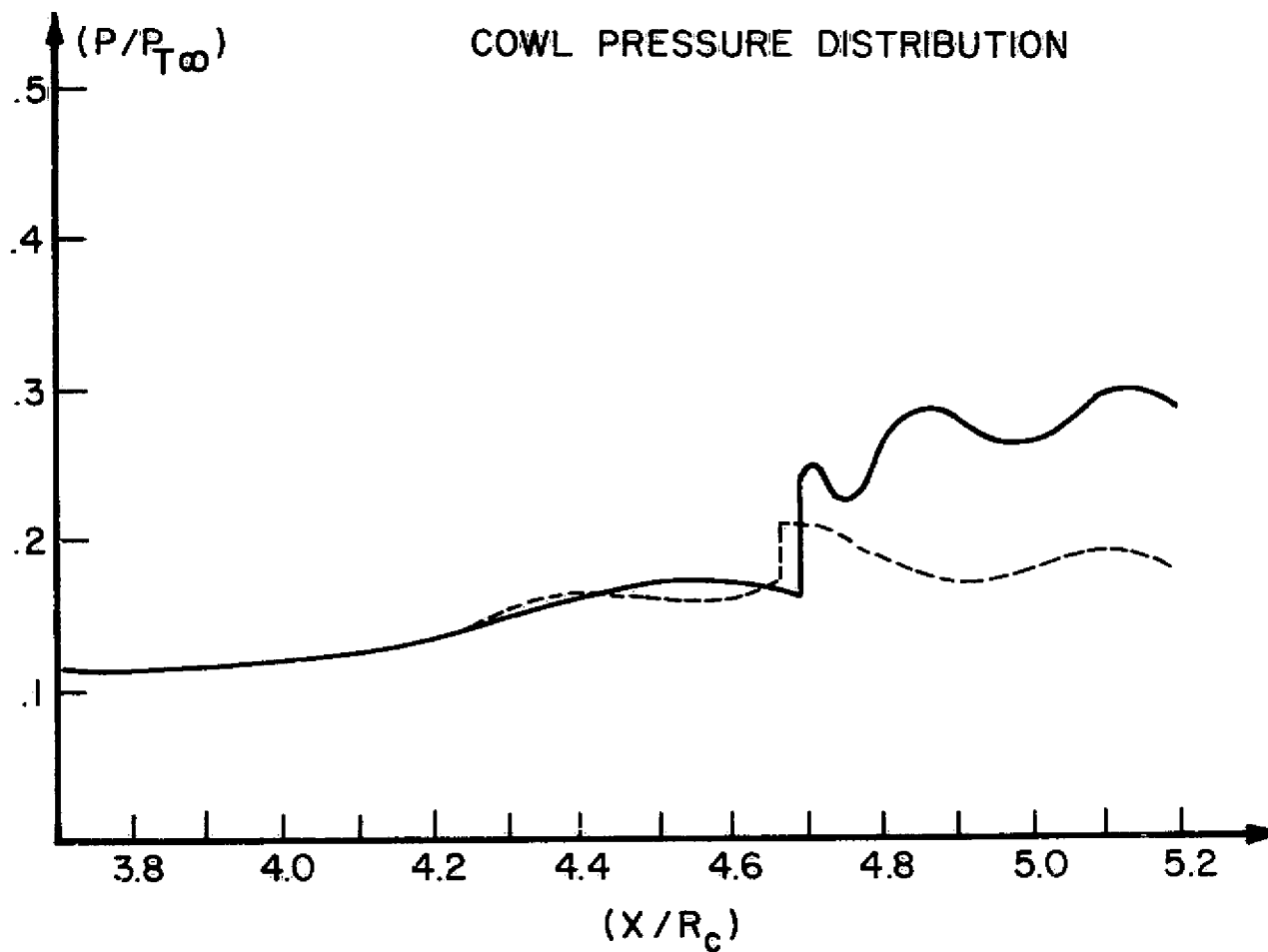


FIGURE 18. (CONTINUED)

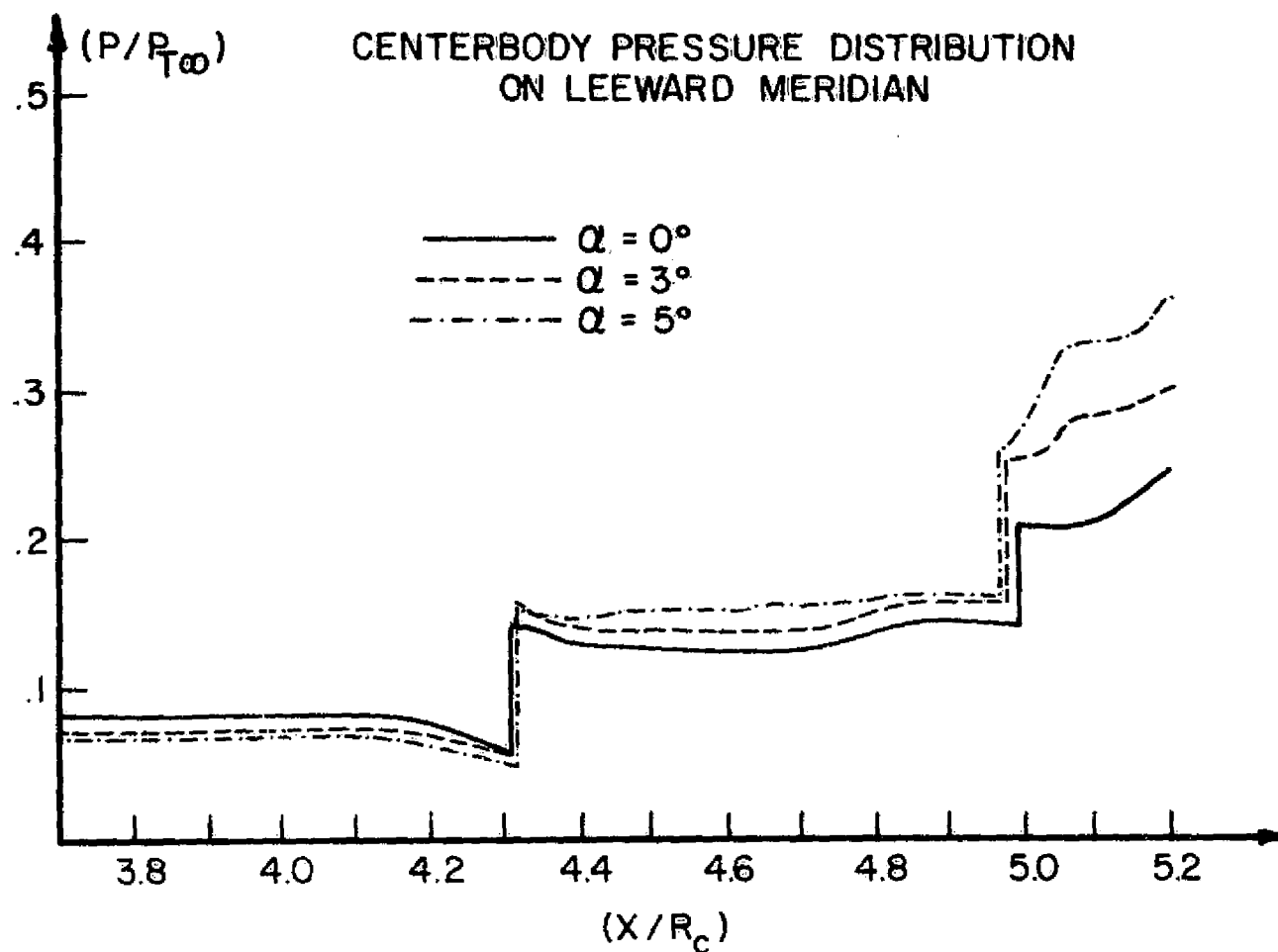


FIGURE 19. COMPUTED CENTERBODY PRESSURE DISTRIBUTIONS
FOR $M_{\infty}=2.5$, $\Delta X/R_c=.855$ WITH $\alpha=0^\circ$, $\alpha=3^\circ$, AND $\alpha=5^\circ$

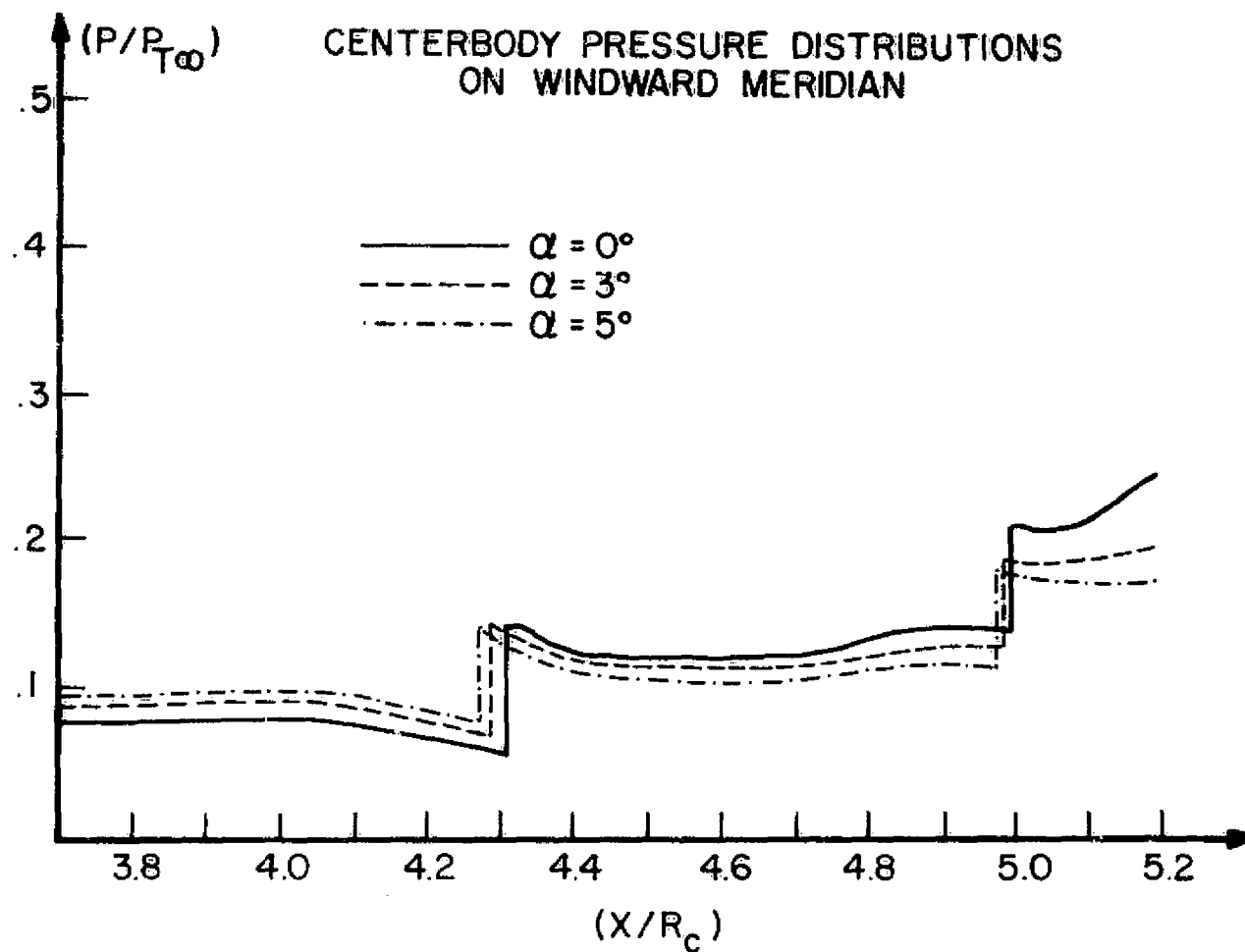


FIGURE 19. (CONTINUED)

computed centerbody and cowl pressure distributions for an incidence angle of $\alpha = 0^\circ$. Although the centerbody has been translated forward, the coordinate system origin is maintained at the forebody tip. Consequently, the internal flow computational regime begins at $x/R_c = 3.715$. Generally speaking, the strength of the internal shock wave system for this case is somewhat reduced as compared to the design point case (see Figure 13). Figure 17 illustrates the computed pressure distributions and some experimental data for an incidence angle of $\alpha = 3.0^\circ$. Pressure distributions for the centerbody and the cowl on both the leeward and the windward meridians are shown. Compared to the $\alpha = 0^\circ$ case, the strength of the internal shock wave system is increased on the leeward side but reduced on the windward side. Experimental data are presented for the centerbody pressure on the leeward meridian and for the cowl pressure on both the leeward and windward meridians. Generally speaking, good overall agreement between theory and experiment is obtained except in regions of high viscous interaction and boundary layer bleed. For all of the three-dimensional computations, 21 circumferential stations and 11 radial stations (9 streamline points and an upstream and downstream shock wave point) were employed in the computed sector (half-plane). The maximum deviation of the mass flow rate at any solution plane from the mass flow rate at the cowl lip solution plane for the $\alpha = 3.0^\circ$ case was 0.44 percent. The computed pressure distributions on the centerbody and the cowl for both the leeward and windward meridians for the incidence angle of $\alpha = 5.0^\circ$ are shown in Figure 18. The leeward meridian shock wave strength has been increased over the $\alpha = 3.0^\circ$ case, whereas the shock wave strength on the windward meridian has been

reduced. The maximum deviation in mass flow rate for the $\alpha = 5.0^\circ$ case was 0.89 percent. Finally, to illustrate the effect of increasing angle of attack on the centerbody pressure distribution, the centerbody results of Figures 16, 17, and 18 are superimposed in Figure 19.

Results for the second off-design case of $M_\infty = 3.3$ and $\Delta x/R_C = 0.356$ are presented in Figures 20 to 22. The computed pressure distributions for the centerbody and the cowl for an incidence angle of $\alpha = 0^\circ$ are presented in Figure 20. With the prescribed centerbody translation, the internal flow computational regime begins at $x/R_C = 3.216$. Figure 21 illustrates the computed centerbody and cowl static pressure distributions on both the leeward and windward meridians for an incidence angle of $\alpha = 3.0^\circ$. The strengthening of the leeward side shock wave and the weakening of the windward side shock wave are again noted. Experimental data for the leeward meridian of the centerbody and for both the leeward and windward meridians of the cowl are also shown in Figure 21. Fairly good overall agreement between theory and experiment is obtained until regions of high viscous interaction and boundary layer removal are reached. Again, 21 circumferential stations and 11 radial stations were used in the computation. The maximum deviation of the mass flow rate at any solution plane compared to that on the cowl lip solution plane for the $\alpha = 3.0^\circ$ case was 0.67 percent. Figure 22 illustrates the computed static pressure distributions for the centerbody and the cowl for an incidence angle of $\alpha = 5.0^\circ$. The maximum deviation in mass flow rate for this case was 0.84 percent.

Finally, comparisons are made between the results of the present analysis and results obtained from the finite difference shock-capturing algorithm developed by Presley (37). At present, the

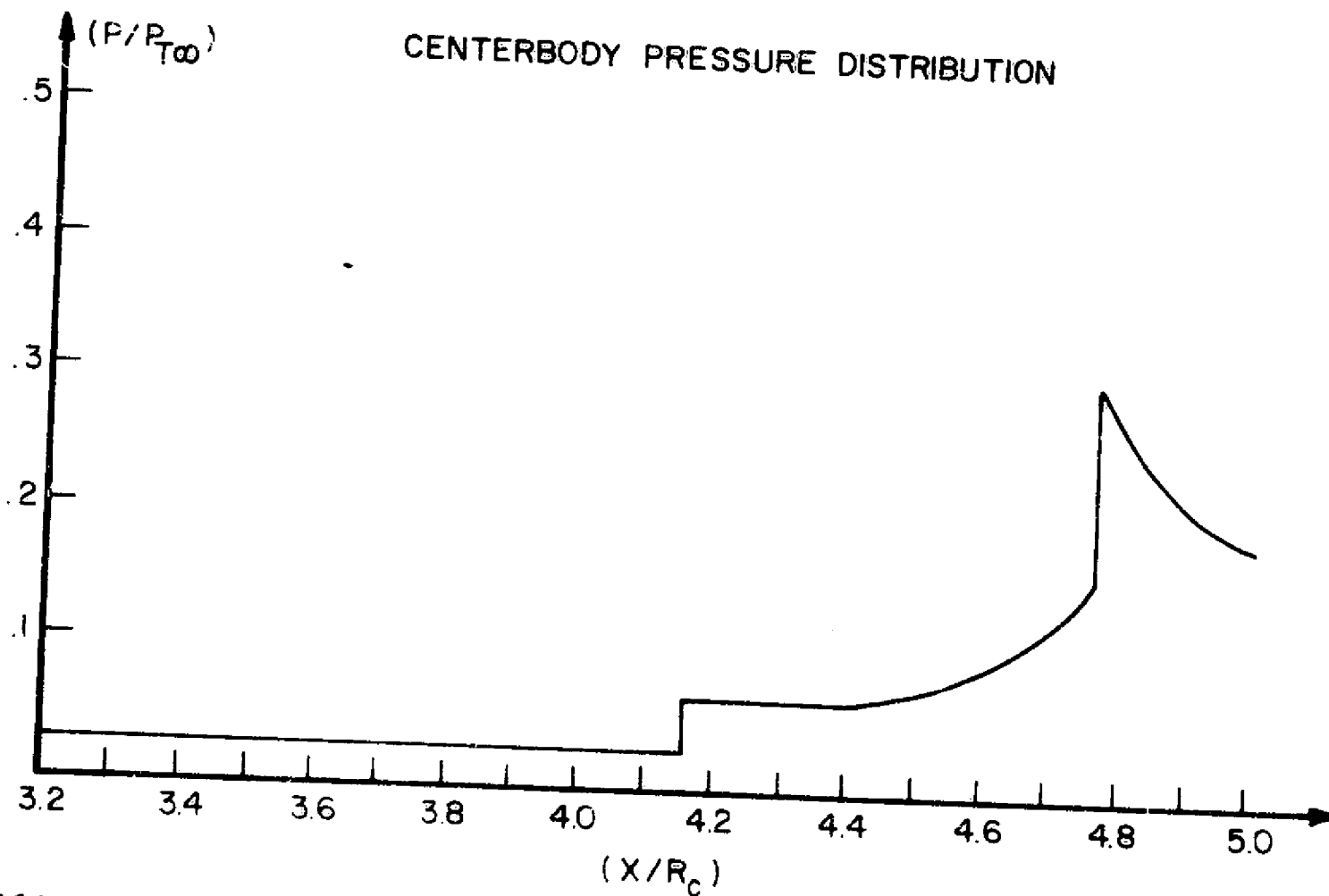


FIGURE 20. THREE-DIMENSIONAL METHOD OF CHARACTERISTICS
SOLUTION FOR $M_\infty = 3.3$, $\alpha = 0^\circ$, $\Delta X/R_c = .356$

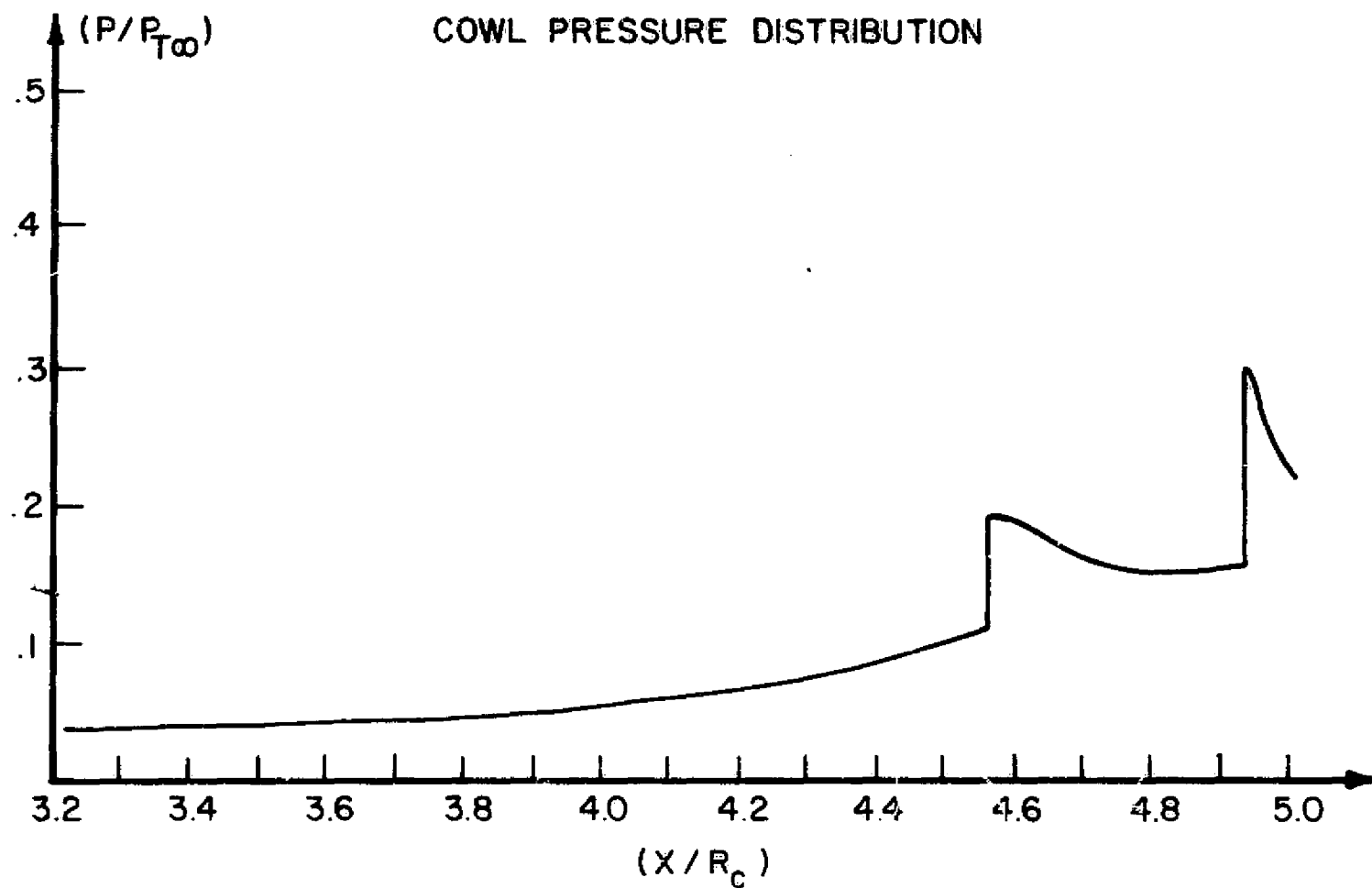


FIGURE 20. (CONTINUED)

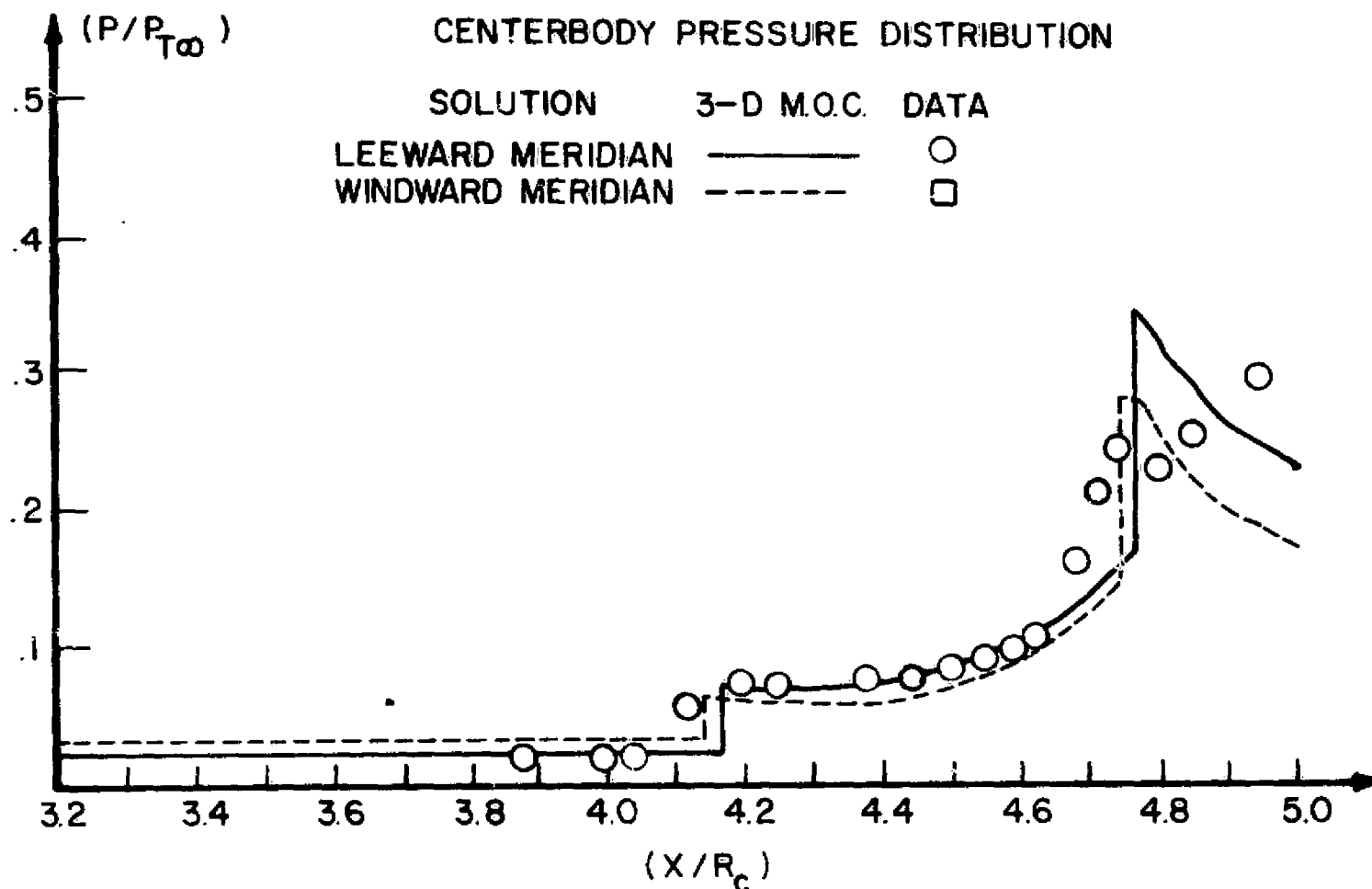


FIGURE 21. COMPARISON OF THREE-DIMENSIONAL METHOD OF CHARACTERISTICS SOLUTION WITH EXPERIMENTAL DATA FOR $M_\infty = 3.3$, $\alpha = 3^\circ$, $\Delta X/R_c = .356$

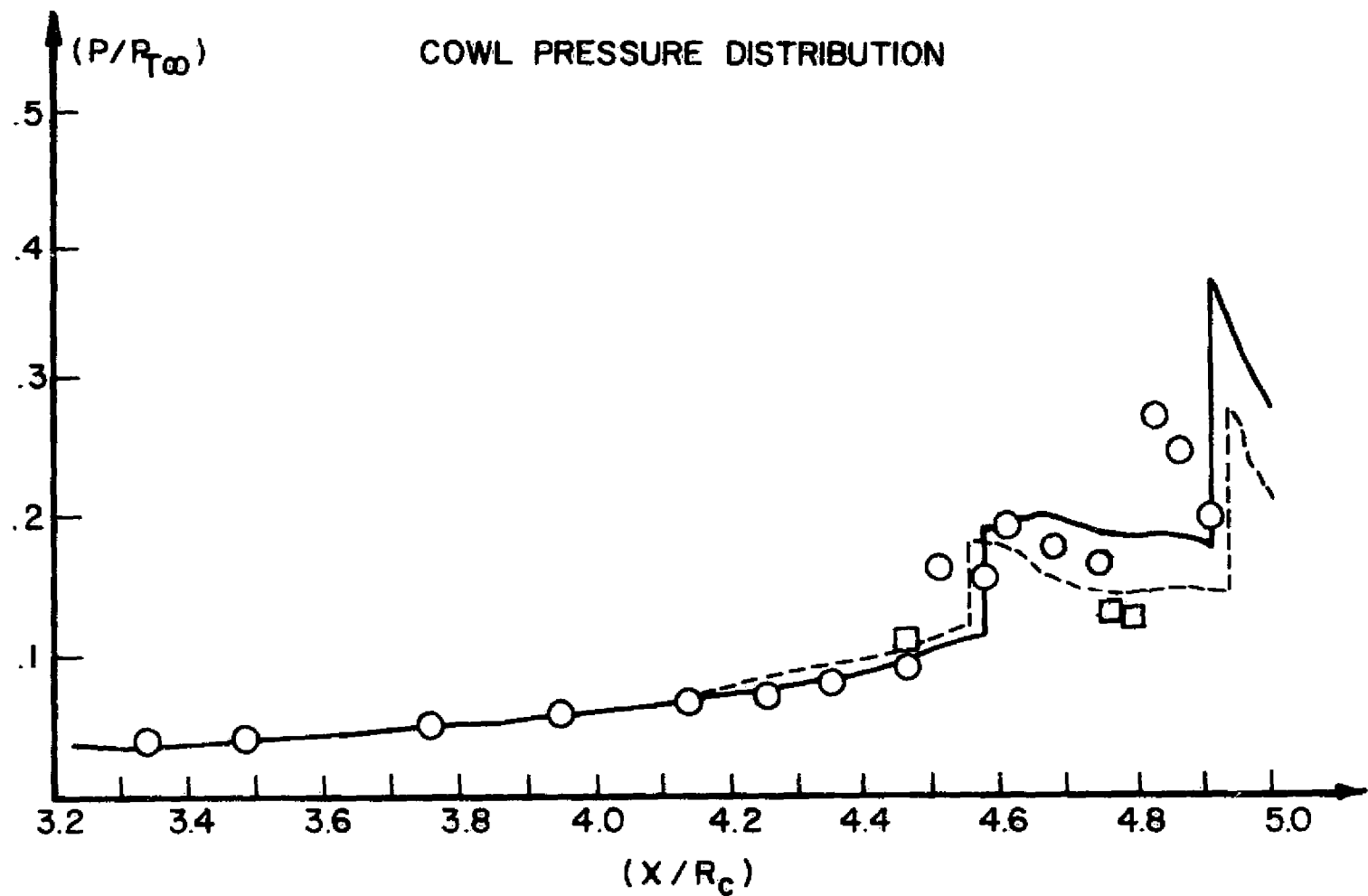


FIGURE 21. (CONTINUED)

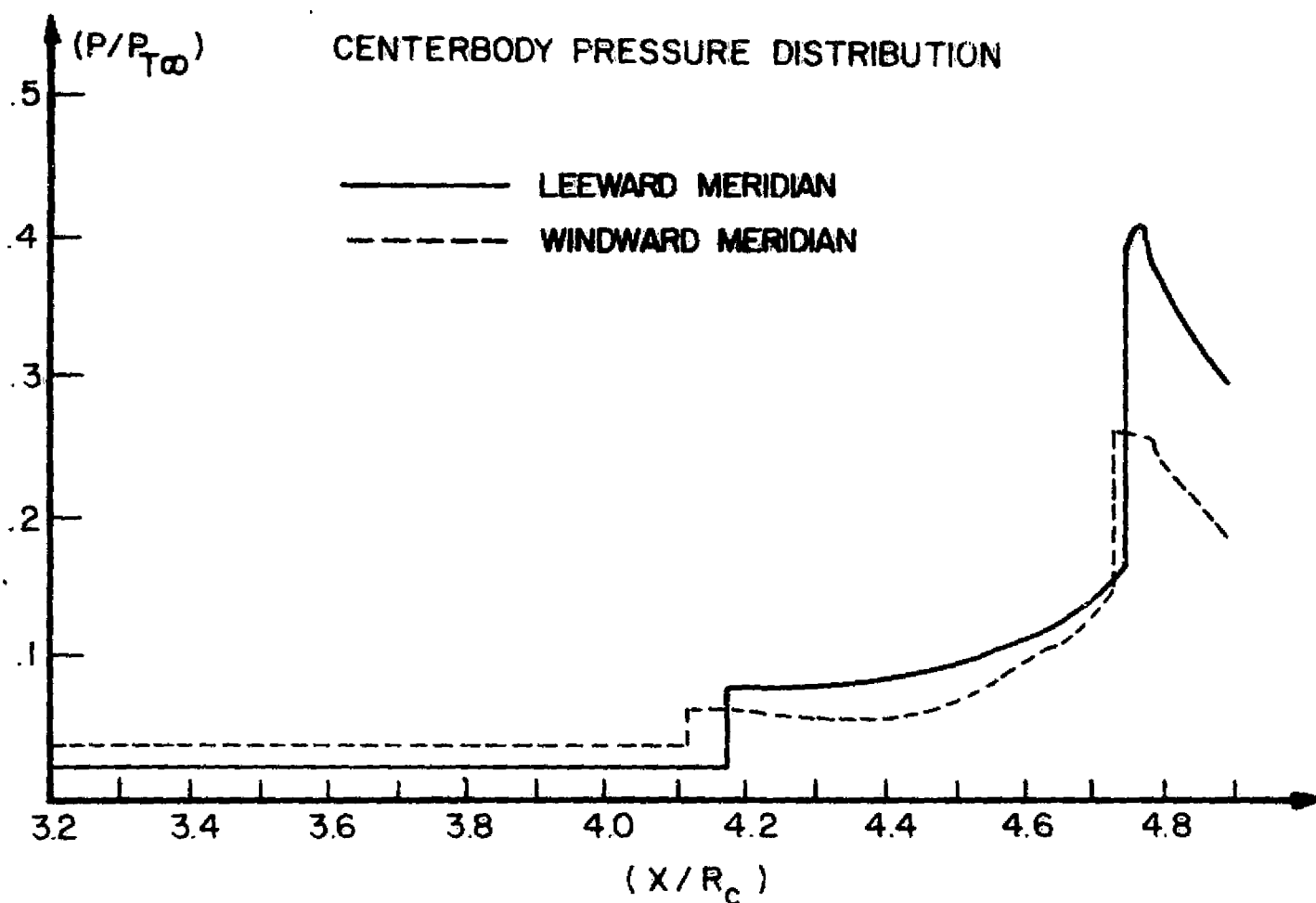


FIGURE 22. THREE-DIMENSIONAL METHOD OF CHARACTERISTICS
SOLUTION FOR $M_\infty = 3.3$, $\alpha = 5^\circ$, $\Delta X/R_c = .356$

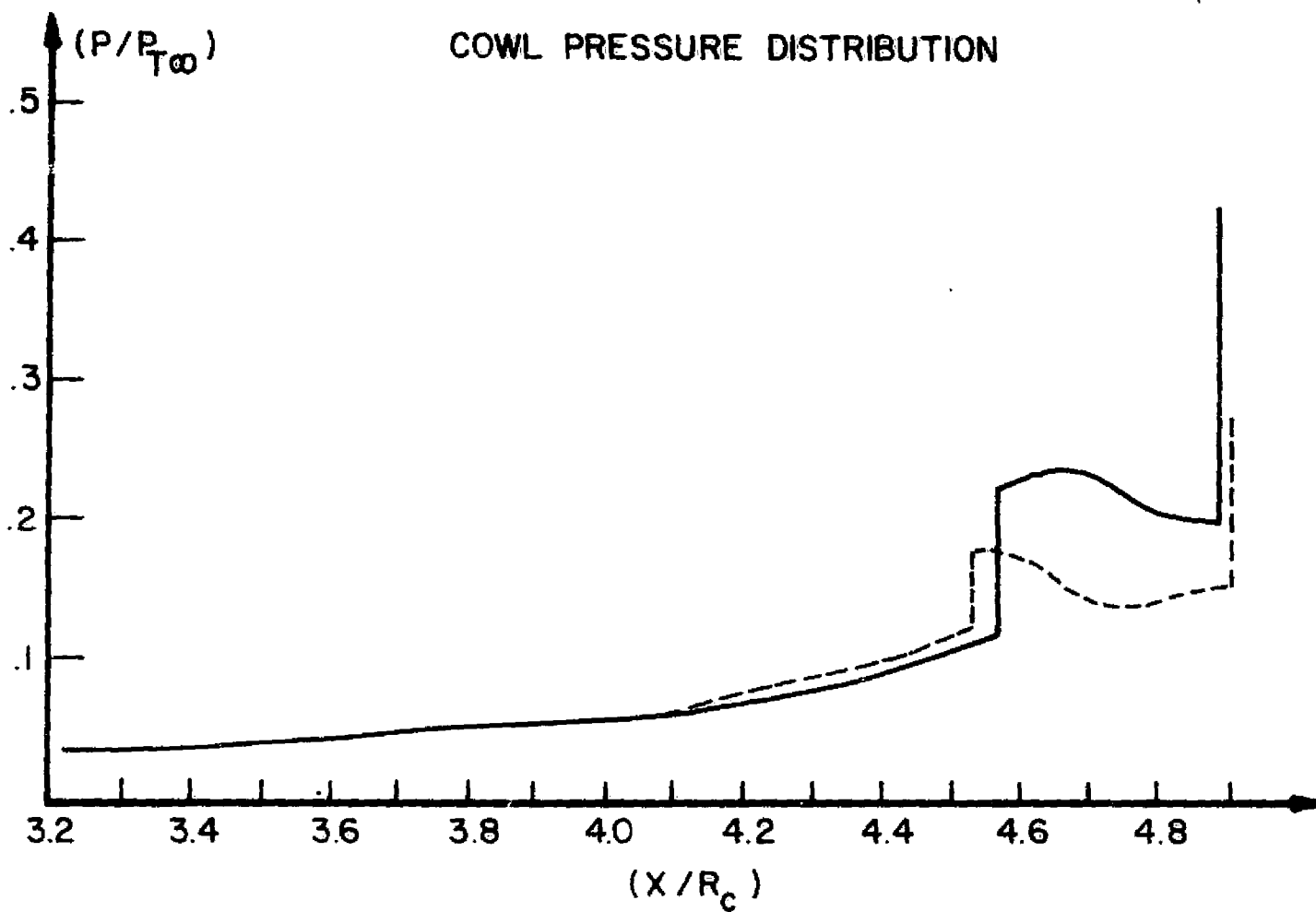


FIGURE 22. (CONTINUED)

computer program developed by Presley is the only other analysis capable of predicting the internal flow field in supersonic mixed-compression inlets at angle of attack. That algorithm employs the second-order accurate finite difference operator devised by MacCormack (38). In that scheme, shock waves are automatically captured in the computational mesh without requiring any special logic which discretely fits discontinuities. The presence of shock waves in the solution is evidenced by steep gradients in the computed flow property fields.

Figure 23 compares the centerbody and cowl pressure distributions obtained by the method of characteristics scheme to those calculated by the shock-capturing technique for the case of $M_\infty = 3.3$, $\alpha = 3.0^\circ$, and $\Delta x/R_c = 0.356$. For the most part, good agreement between the two analyses is obtained. In the method of characteristics solution, however, the shock wave solid boundary intersections are more sharply defined. This result is to be expected, since in the shock-capturing technique shock waves are not discretely fitted but rather are smeared over a number of mesh points. The shock-capturing algorithm employed 11 circumferential stations and 21 radial stations for the solution presented in Figure 23.

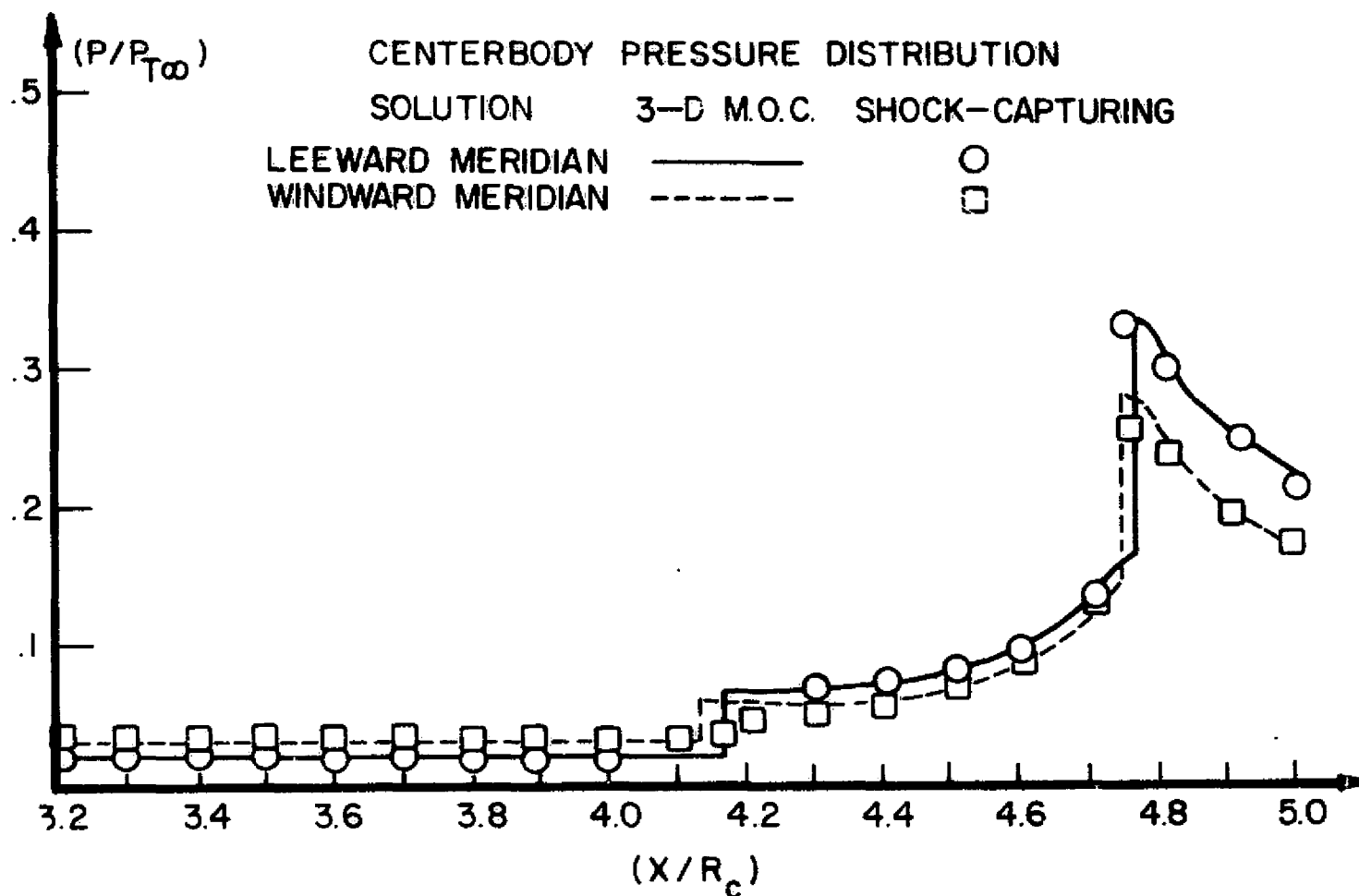


FIGURE 23. COMPARISON OF THE THREE-DIMENSIONAL METHOD OF CHARACTERISTICS WITH A FINITE DIFFERENCE SHOCK-CAPTURING TECHNIQUE FOR $M_\infty = 3.3$, $\alpha = 3^\circ$, $\Delta X/R_c = .356$

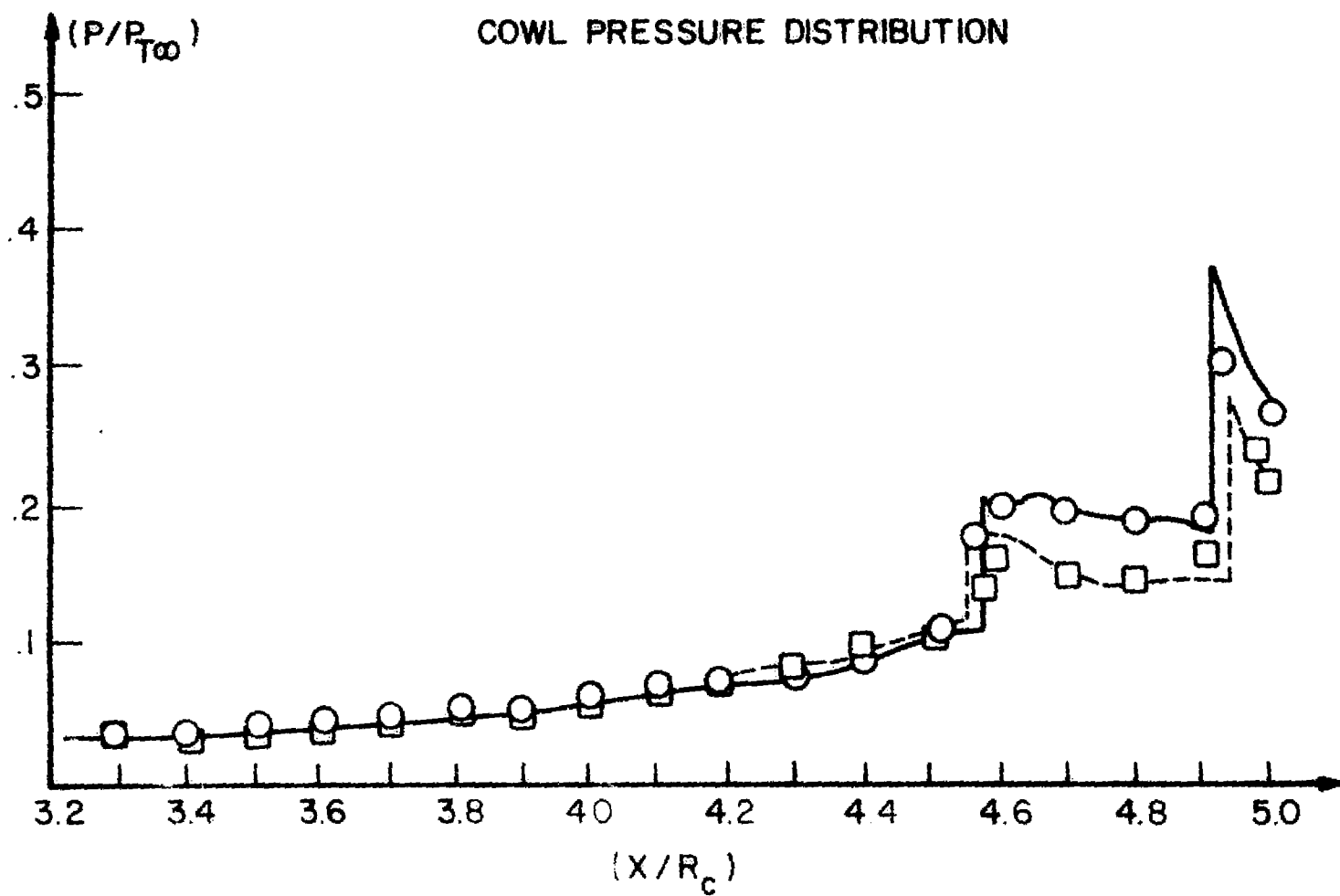


FIGURE 23. (CONTINUED)

SECTION VII

CONCLUSIONS

The flow field in a supersonic mixed-compression aircraft inlet at nonzero angle of attack has been computed using the method of characteristics for steady three-dimensional flow in conjunction with a discrete shock wave fitting procedure. The culmination of the present research effort is a production type computer program which has the capability to predict the flow field in a variety of axisymmetric mixed-compression aircraft inlets. A number of conclusions concerning the present analysis can be made:

1. The external flow field about the forebody can be accurately calculated if a bow shock wave of reasonably strong strength exists.
2. For axisymmetric flows, the solution obtained by the present analysis agrees well with the solution obtained by the two-dimensional method of characteristics.
3. Except in the regions of strong viscous interaction and boundary layer removal, the results of the present analysis agree well with experimental data.
4. Good agreement is obtained between the present analysis and a finite difference shock-capturing technique for three-dimensional flow solutions. The present analysis, however, which discretely fits shock waves, provides a better resolution of the shock wave structure.

5. Without the matching of the present analysis to a higher-order boundary layer analysis, including the influence of molecular transport in the computation has little or no effect on the solution.

Although the inlets analyzed were axisymmetric inlets, the computer program can be readily modified to analyze geometries which have noncircular cross-sections. Moreover, the inclusion of finite rate chemical reactions in the thermodynamic model is reasonably straightforward. The analysis can be modified to compute the external flow about a stepped cone and to compute the internal flow when the bow shock wave has been ingested into the annulus. Perhaps the most worthwhile endeavor, though, would be to mate the present analysis with a three-dimensional compressible turbulent boundary layer analysis. The boundary layer analysis should have well developed three-dimensional turbulence models, an accurate means of computing an oblique shock wave-boundary layer interaction in three-dimensions, and the capability to account for boundary layer removal.

APPENDIX A

GOVERNING EQUATIONS

1. INTRODUCTION

The major assumptions constituting the gas dynamic model are:

1. continuum flow
2. steady flow
3. negligible body forces
4. thermodynamic equilibrium (i.e., mechanical, thermal, and chemical equilibrium)
5. no mass diffusion
6. negligible radiative heat transfer and no internal heat generation other than viscous dissipation
7. viscous and thermal diffusion effects of secondary importance in determining the solution

The governing equations for the assumed flow model consist of the continuity equation, the component momentum equations, the energy equation, the thermal and caloric equations of state, and the appropriate representations for the molecular transport properties. These relations are presented in this appendix

2. DIFFERENTIAL EQUATIONS OF MOTION

The general continuity equation^{*} (29) is

^{*} Repeated indices imply summation over the range of 1 to 3 unless otherwise noted.

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \quad (A.1)$$

where t denotes time, ρ is the density, x_i ($i=1,2,3$) denotes the three rectangular coordinates x , y , and z , respectively, and u_i ($i=1,2,3$) denotes the corresponding velocity components u , v , and w , respectively. The operator $D()/Dt$ in equation (A.1) is the material derivative given by

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + u_j \frac{\partial()}{\partial x_j} \quad (A.2)$$

For steady three-dimensional flow, equation (A.1) may be written in expanded form as

$$\rho u_x + \rho v_y + \rho w_z + u\rho_x + v\rho_y + w\rho_z = 0 \quad (A.3)$$

where the subscripts x , y , and z denote partial differentiation with respect to the corresponding direction.

The appropriate momentum equation is the Navier-Stokes equation (29), which written in component form is

$$\begin{aligned} \rho \frac{Du_i}{Dt} = B_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{2}{3} \frac{\partial}{\partial x_i} \left(\mu \frac{\partial u_j}{\partial x_j} \right) \\ + \frac{\partial}{\partial x_i} \left(\eta \frac{\partial u_j}{\partial x_j} \right) \quad (i=1,2,3) \end{aligned} \quad (A.4)$$

where B_i denotes the i th component of the body force, P is the pressure, μ denotes the dynamic viscosity, and η is the second coefficient of viscosity.

A major assumption of the present analysis is that the effects of viscous and thermal diffusion are of secondary importance in determining the solution as compared to the inertial effects. Consistent with this assumption of inertial dominance, the viscous and thermal diffusion terms in the governing differential equations will be treated as forcing or correction terms in the method of characteristic scheme to be presented. In the following, the viscous and thermal transport terms will be placed on the right-hand sides of the respective governing equations. The convective terms will be placed on the left-hand sides, and will be considered as constituting the principal parts of these equations. Thus, writing equation (A.4) with the assumptions of steady flow, negligible body forces, $\eta = 0$ [Stokes's hypothesis (30)], and inertial dominance gives

$$\rho u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = F_i \quad (i=1,2,3) \quad (A.5)$$

where

$$F_i = \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{2}{3} \frac{\partial}{\partial x_i} \left(\mu \frac{\partial u_j}{\partial x_j} \right) \quad (i=1,2,3) \quad (A.6)$$

Treating the viscosity as a variable, equations (A.5) and (A.6) can be written in expanded form for each of the three coordinate directions as

$$\rho u u_x + \rho v u_y + \rho w u_z + P_x = F_x \quad (A.7)$$

$$\rho u v_x + \rho v v_y + \rho w v_z + P_y = F_y \quad (A.8)$$

$$\rho u w_x + \rho v w_y + \rho w w_z + P_z = F_z \quad (A.9)$$

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where

$$F_x = \mu_x \left[\frac{4}{3} u_x - \frac{2}{3} (v_y + w_z) \right] + \mu_y (u_y + v_x) + \mu_z (u_z + w_x) \\ + \mu \left[\frac{4}{3} u_{xx} + u_{yy} + u_{zz} + \frac{1}{3} (v_{xy} + w_{xz}) \right] \quad (A.10)$$

$$F_y = \mu_y \left[\frac{4}{3} v_y - \frac{2}{3} (u_x + w_z) \right] + \mu_x (v_x + u_y) + \mu_z (v_z + w_y) \\ + \mu \left[\frac{4}{3} v_{yy} + v_{xx} + v_{zz} + \frac{1}{3} (u_{yx} + w_{yz}) \right] \quad (A.11)$$

$$F_z = \mu_z \left[\frac{4}{3} w_z - \frac{2}{3} (u_x + v_y) \right] + \mu_x (w_x + u_z) + \mu_y (w_y + v_z) \\ + \mu \left[\frac{4}{3} w_{zz} + w_{xx} + w_{yy} + \frac{1}{3} (u_{zx} + v_{zy}) \right] \quad (A.12)$$

Finally, it remains to obtain an appropriate form of the energy equation. It is assumed in the present analysis that the working gas can be represented as a simple system in thermodynamic equilibrium. Under this assumption the thermodynamic relation (31)

$$Tds = dh - \frac{dP}{\rho} \quad (A.13)$$

is valid, where T denotes the absolute temperature, s is the entropy per unit mass, and h is the enthalpy per unit mass. For a simple system, specification of any two independent thermodynamic properties defines the thermodynamic state of the system (31). Thus,

$$P = P(\rho, s) \quad (A.14)$$

Employing the concept of the total derivative, and introducing the material derivative operator given by equation (A.2), the following relation may be obtained from equation (A.14).

$$\frac{DP}{Dt} = \left(\frac{\partial P}{\partial \rho} \right)_s \frac{D\rho}{Dt} + \left(\frac{\partial P}{\partial s} \right)_\rho \frac{Ds}{Dt} \quad (\text{A.15})$$

The sonic speed a is defined by

$$a^2 = \left(\frac{\partial P}{\partial \rho} \right)_s \quad (\text{A.16})$$

Thus, equation (A.15) may be written as

$$\frac{DP}{Dt} - a^2 \frac{D\rho}{Dt} = F_e \quad (\text{A.17})$$

where

$$F_e = \left(\frac{\partial P}{\partial s} \right)_\rho \frac{Ds}{Dt} \quad (\text{A.18})$$

The material derivative of entropy in equation (A.18) may be expressed in terms of a thermal conduction function and a viscous dissipation function. Consider the energy equation in the following form (29).

$$\rho \frac{De}{Dt} = \frac{\partial}{\partial x_i} \left(\kappa \frac{\partial T}{\partial x_i} \right) + \frac{P}{\rho} \frac{D\rho}{Dt} + \phi \quad (\text{A.19})$$

In equation (A.19), e denotes the internal energy per unit mass, κ is the thermal conductivity, and ϕ represents the viscous dissipation function which for $\eta = 0$ is given by

$$\phi = \frac{1}{2} \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right)^2 \quad (\text{A.20})$$

where δ_{ij} is the Kronecker delta. Using the definition of enthalpy ($h = e + P/\rho$) in equation (A.13) yields

$$Tds = de - \frac{P}{\rho} d\rho \quad (A.21)$$

From equation (A.21) the material derivative of internal energy may be written as

$$\frac{De}{Dt} = T \frac{Ds}{Dt} + \frac{P}{\rho} \frac{D\rho}{Dt} \quad (A.22)$$

Introducing equation (A.22) into equation (A.19) yields

$$\rho T \frac{Ds}{Dt} = \frac{\partial}{\partial x_i} \left[\kappa \frac{\partial T}{\partial x_i} \right] + \phi \quad (A.23)$$

Substituting equation (A.23) into equation (A.18) gives

$$F_e = \xi \left[\frac{\partial}{\partial x_i} \left[\kappa \frac{\partial T}{\partial x_i} \right] + \phi \right] \quad (A.24)$$

where

$$\xi = \frac{1}{\rho T} \left(\frac{\partial P}{\partial s} \right)_\rho \quad (A.25)$$

By treating the thermal conductivity as a variable, and assuming steady three-dimensional flow, equations (A.17) and (A.24) may be written as

$$uP_x + vP_y + wP_z - a^2(u\rho_x + v\rho_y + w\rho_z) = F_e \quad (A.26)$$

where

$$\begin{aligned} F_e = \xi \left\{ \kappa (T_{xx} + T_{yy} + T_{zz}) + \kappa_x T_x + \kappa_y T_y + \kappa_z T_z \right. \\ \left. + \mu \left[2 \left(u_x^2 + v_y^2 + w_z^2 + u_y v_x + u_z w_x + v_z w_y \right) + v_x^2 + w_x^2 \right. \right. \\ \left. \left. + u_y^2 + w_y^2 + u_z^2 + v_z^2 - \frac{2}{3} (u_x + v_y + w_z)^2 \right] \right\} \quad (A.27) \end{aligned}$$

As in the component momentum equations, the viscous and thermal diffusion terms in the energy equation have been placed on the right-hand side and will be treated as forcing functions in the method of characteristics scheme to be presented. The left-hand side is composed of the convective terms which are considered to constitute the principal part of this equation.

3. THERMODYNAMIC MODEL

Before a solution to the system of governing partial differential equations can be obtained, the temperature T , sonic speed a , thermodynamic parameter ξ , viscosity μ , and thermal conductivity κ must be expressed in terms of the dependent variables P and ρ . The representations for T , a , and ξ are discussed in this section. The relations for μ and κ are presented in the next section.

The general functional forms of the temperature T , sonic speed a , and thermodynamic parameter ξ may be expressed as

$$T = T(P, \rho) \quad (\text{A.28})$$

$$a = a(P, \rho) \quad (\text{A.29})$$

$$\xi = \xi(P, \rho) \quad (\text{A.30})$$

For multicomponent systems, with either frozen or equilibrium chemical composition, the functional relationships for T , a , and ξ are obtained from thermochemical calculations. In the case of a thermally and calorically perfect gas, the functional relationships for T , a , and ξ are simple analytical expressions given by

$$T = P/\rho R \quad (A.31)$$

$$a = (\gamma P/\rho)^{1/2} \quad (A.32)$$

$$\xi = \gamma - 1 \quad (A.33)$$

where γ is the specific heat ratio, and R is the gas constant.

In the computer program developed in the present investigation, the temperature, sonic speed, and thermodynamic parameter ξ are calculated in a separate subroutine. The assumed thermodynamic model is that of a thermally and calorically perfect gas, thus, equations (A.31) to (A.33) are employed. Substitution of a replacement subroutine for the existing one allows other thermodynamic models to be specified.

4. TRANSPORT PROPERTIES

Representations are required for the viscosity, the thermal conductivity, and their spatial gradients. Both viscosity and thermal conductivity are functions of temperature and pressure. Hence,

$$\mu = \mu(T, P) \quad (A.34)$$

$$\kappa = \kappa(T, P) \quad (A.35)$$

Using equations (A.34) and (A.35), the spatial derivatives of viscosity and thermal conductivity may be written as

$$\frac{\partial \mu}{\partial x_i} = \left(\frac{\partial \mu}{\partial T} \right)_P \frac{\partial T}{\partial x_i} + \left(\frac{\partial \mu}{\partial P} \right)_T \frac{\partial P}{\partial x_i} \quad (A.36)$$

$$\frac{\partial \kappa}{\partial x_i} = \left(\frac{\partial \kappa}{\partial T} \right)_P \frac{\partial T}{\partial x_i} + \left(\frac{\partial \kappa}{\partial P} \right)_T \frac{\partial P}{\partial x_i} \quad (A.37)$$

Hence, spatial derivatives of pressure and temperature are also required. Spatial derivatives of pressure and density are employed in the basic integration scheme (even for the inviscid flow case). Thus, those derivatives are already available. Spatial derivatives of temperature can be expressed in terms of spatial derivatives of pressure and density by differentiating the thermal equation of state, equation (A.28).

The pressure dependency indicated in equations (A.34) and (A.35) is usually quite weak, and often both the viscosity and the thermal conductivity are assumed to be functions of temperature only. Thus,

$$\mu = \mu(T) \quad (\text{A.38})$$

$$\kappa = \kappa(T) \quad (\text{A.39})$$

The Sutherland formula (30) is a good representation for equation (A.38).

$$\mu = \mu_0 \left(\frac{T}{T_0} \right)^{1.5} \left(\frac{T_0 + S}{T + S} \right) \quad (\text{A.40})$$

In equation (A.40), μ_0 is the viscosity at the reference temperature T_0 , and S is a constant. Equation (A.39) can be represented by the quadratic expression

$$\kappa = a_1 + a_2 T + a_3 T^2 \quad (\text{A.41})$$

where the coefficients a_i ($i=1,2,3$) are obtained by curve fitting thermal conductivity data.

In the computer program, the viscosity, the thermal conductivity, and their spatial derivatives are computed in a separate subroutine.

The assumed functional forms of viscosity and thermal conductivity are given by equations (A.40) and (A.41), respectively. The coefficients a_i ($i=1,2,3$) in equation (A.41) are internally generated in the computer program by curve fitting user supplied data. Different formulations for the transport properties can be implemented into the computer program by supplying an appropriate replacement subroutine.

APPENDIX B

DERIVATION OF THE EQUATIONS FOR THE CHARACTERISTIC
SURFACES AND THE COMPATIBILITY RELATIONS

1. INTRODUCTION

Systems of hyperbolic partial differential equations in n independent variables have the property that there exist surfaces in n -space on which linear combinations of the original differential equations can be formed that contain derivatives only in the surfaces themselves. Differentiation in these surfaces is performed in $(n-1)$ -space. The resulting differential operators are interior operators which are known as compatibility relations. The surfaces are called characteristic surfaces. A compatibility relation is valid only when it is applied on its corresponding characteristic surface. Furthermore, data cannot be arbitrarily specified on a characteristic surface, but instead must satisfy the compatibility relation.

The method of characteristics is based on replacing the original system of partial differential equations with an equivalent number of compatibility relations applied on the appropriate characteristic surfaces. In flows with two independent variables, the method of characteristics has the advantage of reducing the solution of a system of partial differential equations to the solution of a system of ordinary differential equations. In three-dimensional flow, however, the resulting compatibility relations are still partial differential equations in two independent directions.

In this appendix, the equations for the characteristic surfaces and the corresponding compatibility relations are derived for steady three-dimensional flow. For a complete discussion of hyperbolic partial differential equations in three independent variables, refer to Courant and Hilbert (39). An excellent presentation of the method of characteristics for three-dimensional flow is given in Zucrow and Hoffman (5).

2. EQUATIONS OF MOTION

The partial differential equations of motion for steady three-dimensional flow consist of the three component momentum equations, the continuity equation, and the energy equation. Those equations are developed in Appendix A, and are repeated below for reference.

$$\rho uu_x + \rho vu_y + \rho wu_z + P_x = F_x \quad (B.1)$$

$$\rho uv_x + \rho vv_y + \rho wv_z + P_y = F_y \quad (B.2)$$

$$\rho uw_x + \rho vw_y + \rho ww_z + P_z = F_z \quad (B.3)$$

$$\rho u_x + \rho v_y + \rho w_z + u\rho_x + v\rho_y + w\rho_z = 0 \quad (B.4)$$

$$uP_x + vP_y + wP_z - a^2(u\rho_x + v\rho_y + w\rho_z) = F_e \quad (B.5)$$

In equations (B.1) to (B.5), u , v , and w denote the x , y , and z components of velocity, respectively, ρ is the density, P is the pressure, a is the sonic speed, and the subscripts x , y , and z denote partial differentiation in the corresponding direction. The nonhomogeneous terms F_x , F_y , F_z , and F_e are the forcing terms in the x , y , and z component momentum equations and the energy equation, respectively. Written in this form, with the left-hand sides constituting the principal

parts, equations (B.1) to (B.5) may be classified as a system of quasi-linear nonhomogeneous partial differential equations of first order. The system is hyperbolic (i.e., has real characteristic surfaces) if the flow is supersonic.

3. CHARACTERISTIC SURFACES

The general compatibility relation, which is a linear combination of the governing partial differential equations, is formed by multiplying equations (B.1) to (B.5) by the arbitrary variables ω_i ($i=1$ to 5), respectively, and summing. This yields

$$\begin{aligned} & \omega_1(\rho uu_x + \rho vu_y + \rho wu_z + P_x) + \omega_2(\rho uv_x + \rho vv_y + \rho vw_z + P_y) \\ & + \omega_3(\rho uw_x + \rho vw_y + \rho ww_z + P_z) + \omega_4(\rho u_x + \rho v_y + \rho w_z \\ & + u\rho_x + v\rho_y + w\rho_z) + \omega_5[uP_x + vP_y + wP_z \\ & - a^2(u\rho_x + v\rho_y + w\rho_z)] = \omega_1 F_x + \omega_2 F_y + \omega_3 F_z + \omega_5 F_e \quad (B.6) \end{aligned}$$

Equation (B.6) may be written as

$$\begin{aligned} & \rho(u\omega_1 + \omega_4)u_x + \rho v\omega_1 u_y + \rho w\omega_1 u_z + \rho u\omega_2 v_x + \rho(v\omega_2 + \omega_4)v_y \\ & + \rho w\omega_2 v_z + \rho u\omega_3 w_x + \rho v\omega_3 w_y + \rho(w\omega_3 + \omega_4)w_z \\ & + (\omega_1 + u\omega_5)P_x + (\omega_2 + v\omega_5)P_y + (\omega_3 + w\omega_5)P_z \\ & + u(\omega_4 - a^2\omega_5)\rho_x + v(\omega_4 - a^2\omega_5)\rho_y + w(\omega_4 - a^2\omega_5)\rho_z \\ & = \omega_1 F_x + \omega_2 F_y + \omega_3 F_z + \omega_5 F_e \quad (B.7) \end{aligned}$$

By noting the coefficients of the partial derivatives in equation (B.7), the following vectors may be defined.

$$\bar{W}_1 = [\rho(u\omega_1 + \omega_4), \rho v\omega_1, \rho w\omega_1] \quad (B.8)$$

$$\bar{W}_2 = [\rho u\omega_2, \rho(v\omega_2 + \omega_4), \rho w\omega_2] \quad (B.9)$$

$$\bar{W}_3 = [\rho u\omega_3, \rho v\omega_3, \rho(w\omega_3 + \omega_4)] \quad (B.10)$$

$$\bar{W}_4 = [(\omega_1 + u\omega_5), (\omega_2 + v\omega_5), (\omega_3 + w\omega_5)] \quad (B.11)$$

$$\bar{W}_5 = [u(\omega_4 - a^2\omega_5), v(\omega_4 - a^2\omega_5), w(\omega_4 - a^2\omega_5)] \quad (B.12)$$

The directional derivative of a function f in some direction

$\bar{l} = (l_x, l_y, l_z)$ is given by

$$\frac{df}{d\bar{l}} = l_x \frac{\partial f}{\partial x} + l_y \frac{\partial f}{\partial y} + l_z \frac{\partial f}{\partial z} \quad (B.13)$$

By considering equations (B.8) to (B.13), equation (B.7) may be written as

$$\frac{du}{dW_1} + \frac{dv}{dW_2} + \frac{dw}{dW_3} + \frac{dP}{dW_4} + \frac{d\rho}{dW_5} = \omega_1 F_x + \omega_2 F_y + \omega_3 F_z + \omega_5 F_e \quad (B.14)$$

where du/dW_1 is the directional derivative of u in the $\bar{W}_1$ direction, etc.

On a characteristic surface, equation (B.14) reduces to an interior operator, that is, differentiation takes place in the surface itself.

For this to occur, the vectors $\bar{W}_i$ ($i=1$ to 5) must all lie in the elemental plane which is tangent to the characteristic surface at the point in consideration. This means that the vectors $\bar{W}_i$ ($i=1$ to 5) are

linearly dependent. Let the normal to the characteristic surface be denoted by $\bar{N} = (N_x, N_y, N_z)$. Hence, on the characteristic surface

$$\bar{N} \cdot \bar{W}_i = 0 \quad (i=1 \text{ to } 5) \quad (\text{B.15})$$

Equation (B.15) yields five linear homogeneous equations which may be written in matrix form as follows

$$\begin{bmatrix} \rho U & 0 & 0 & \rho N_x & 0 \\ 0 & \rho U & 0 & \rho N_y & 0 \\ 0 & 0 & \rho U & \rho N_z & 0 \\ N_x & N_y & N_z & 0 & U \\ 0 & 0 & 0 & U & -a^2 U \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{bmatrix} = 0 \quad (\text{B.16})$$

where

$$U = uN_x + vN_y + wN_z \quad (\text{B.17})$$

Since the system given by equation (B.16) is homogeneous, a nontrivial solution exists only if the coefficient matrix is singular, which means its determinant must be zero. Evaluating the determinant and equating it to zero yields

$$(\rho U)^3 [U^2 - a^2(N_x^2 + N_y^2 + N_z^2)] = 0 \quad (\text{B.18})$$

Equation (B.18) is the characteristic equation for the original system of equations, equations (B.1) to (B.5). The form of equation (B.18) is that of a repeated linear factor and a quadratic factor.

Equating the two factors in equation (B.18) to zero yields the equations of two real nonintersecting cones formed by the envelope of the characteristic normals at a point. Setting the linear factor in equation (B.18) to zero gives (the case of $\rho = 0$ is immediately dismissed)

$$uN_x + vN_y + wN_z = 0 \quad (B.19)$$

Equation (B.19) represents a degenerate cone formed by the envelope of characteristic normals at a point, each normal being orthogonal to the local velocity vector. Hence, equation (B.19) represents a plane normal to a streamline. The characteristic surface is the reciprocal cone to this degenerate cone of normals, and, hence, is also degenerate, consisting of line segments tangent to the streamlines. Characteristic surfaces with normal components satisfying equation (B.19) are called stream surfaces. The envelope of all stream surfaces at a point is a single pencil of planes whose axis is a streamline. A streamline may be represented by the following equations

$$dx/dt = u \quad dy/dt = v \quad dz/dt = w \quad (B.20)$$

where t is the time of travel of a fluid particle along the streamline.

Equating the quadratic factor in equation (B.18) to zero gives

$$(uN_x + vN_y + wN_z)^2 - a^2(N_x^2 + N_y^2 + N_z^2) = 0 \quad (B.21)$$

Equation (B.21) represents the quadric surface of a right circular cone formed by the envelope of characteristic normals at a point. In gas dynamics this cone is usually referred to as the cone of normals, and is a real cone if $q > a$, where q is the velocity magnitude. Equation (B.21) may be written as

$$un_x + vn_y + wn_z = a \quad (B.22)$$

where $\hat{n} = (n_x, n_y, n_z)$ is the unit normal to the characteristic surface. Equation (B.22) was obtained by arbitrarily selecting the positive root, and the results which follow are consistent with that selection. Characteristic surfaces whose normal components satisfy equation (B.21), or equation (B.22), are called wave surfaces.

Equation (B.21) is the equation for the cone of normals, which is a quadric surface. In general, a quadric surface may be expressed as^{*}

$$A_{ij} dx_i dx_j = 0 \quad (B.23)$$

where x_i ($i=1,2,3$) denotes the three cartesian coordinates x , y , and z , respectively, and A is a nine element coefficient matrix of order two. A normal vector is a directed line segment, so

$$N_i = \sigma dx_i \quad (i=1,2,3) \quad (B.24)$$

where N_i is the i th component of the normal vector, and σ is a constant proportional to the length of the normal. By considering equations (B.23) and (B.24), equation (B.21) may be written as

$$(u_i u_j - a^2 \delta_{ij}) dx_i dx_j = 0 \quad (B.25)$$

where u_i ($i=1,2,3$) denotes the three velocity components u , v , and w , respectively, and δ_{ij} is the Kronecker delta.

* Repeated indices imply summation over the range of 1 to 3 unless otherwise noted.

The characteristic cone, which is the envelope of all wave surfaces at a point, is the reciprocal cone to the cone of normals given by equation (B.21), or equation (B.25). The geometrical relationship between these surfaces is shown in Figure B.1. If the general form of the equation of the cone of normals is given by equation (B.23), then the reciprocal cone is given by (9)

$$A_{ij}^{-1} dx_i dx_j = 0 \quad (B.26)$$

where A^{-1} is the inverse of the nine element symmetric matrix A in equation (B.23). Using equation (B.25) to determine A from which A^{-1} may be determined, equation (B.26) for the characteristic cone may be written as

$$[u_i u_j - (q^2 - a^2) \delta_{ij}] dx_i dx_j = 0 \quad (B.27)$$

Equation (B.27) represents a real cone if $q > a$. Writing equation (B.27) in expanded form yields

$$\begin{aligned} [u^2 - (q^2 - a^2)] dx^2 + [v^2 - (q^2 - a^2)] dy^2 + [w^2 - (q^2 - a^2)] dz^2 \\ + 2uv(dx)(dy) + 2uw(dx)(dz) + 2vw(dy)(dz) = 0 \end{aligned} \quad (B.28)$$

The characteristic cone given by equation (B.28) is known as the Mach cone and represents the envelope of all wave surfaces at a point. The line of tangency between a particular wave surface and the Mach cone is known as a bicharacteristic. Integration of equation (B.28) gives the curved cone known as the Mach conoid.

In summary, for steady three-dimensional flow there are two families of characteristic surfaces: stream surfaces and wave surfaces

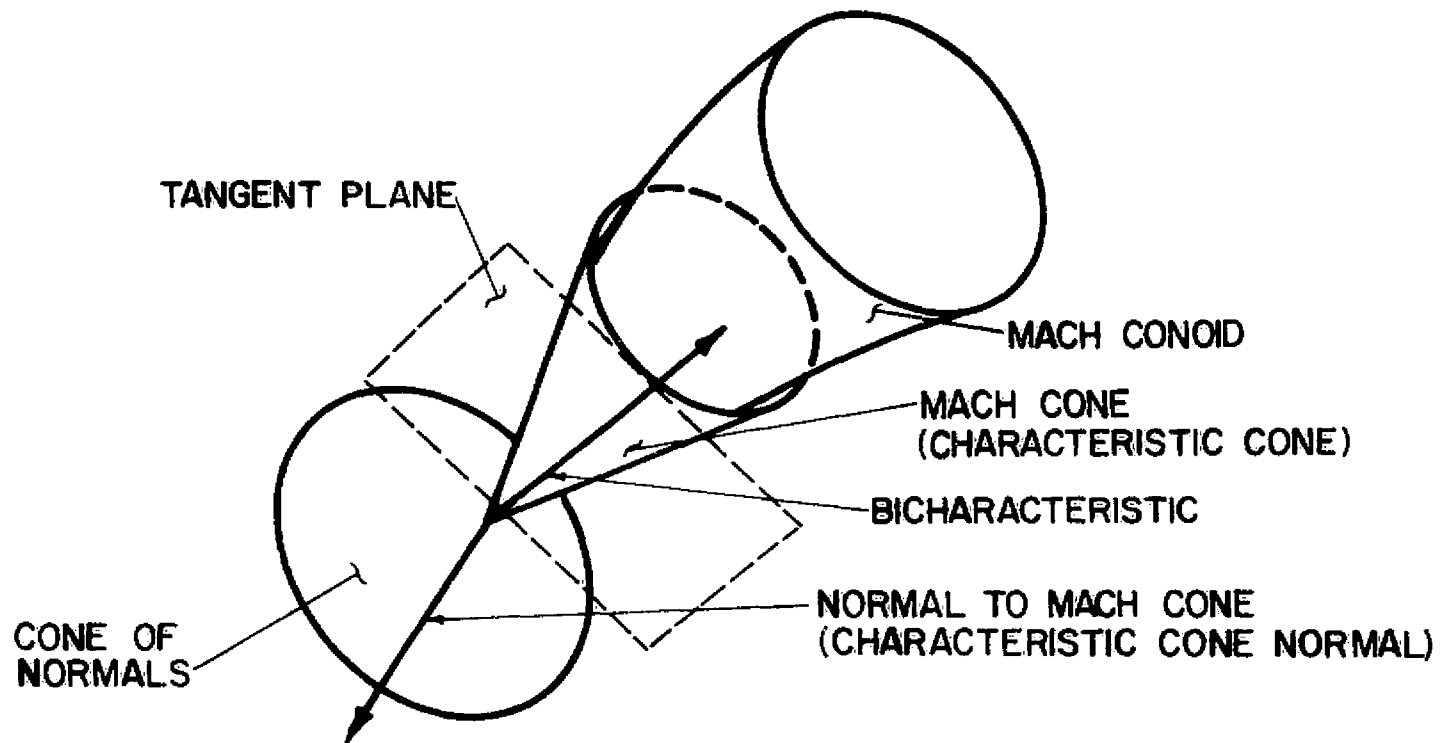


FIGURE B.1. RELATION BETWEEN THE CONE OF NORMALS AND THE CHARACTERISTIC CONE (MACH CONE)

(see Figure B.2). The normal to a stream surface must satisfy equation (B.19), and, hence, the stream surface contains the local velocity vector. The envelope of all stream surfaces at a point is the streamline through the point. The normal to a wave surface must satisfy equation (B.21). The envelope of all wave surfaces at a point is the Mach cone. The line of contact between a particular wave surface and the Mach cone is called a bicharacteristic. At any point there are an infinite number of stream surfaces and wave surfaces.

4. SOLUTION FOR THE ω_i

On a characteristic surface, equation (B.14) reduces to an interior operator, that is, it becomes a compatibility relation. To obtain the exact form of the compatibility relation, the ω_i ($i=1$ to 5) must be determined.

For a stream surface, equation (B.19), repeated below, is valid.

$$uN_x + vN_y + wN_z = U = 0 \quad (B.19)$$

Substitution of equation (B.19) into the homogeneous system given by equation (B.16) yields

$$\begin{bmatrix} 0 & 0 & 0 & \rho N_x & 0 \\ 0 & 0 & 0 & \rho N_y & 0 \\ 0 & 0 & 0 & \rho N_z & 0 \\ N_x & N_y & N_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{bmatrix} = 0 \quad (B.29)$$

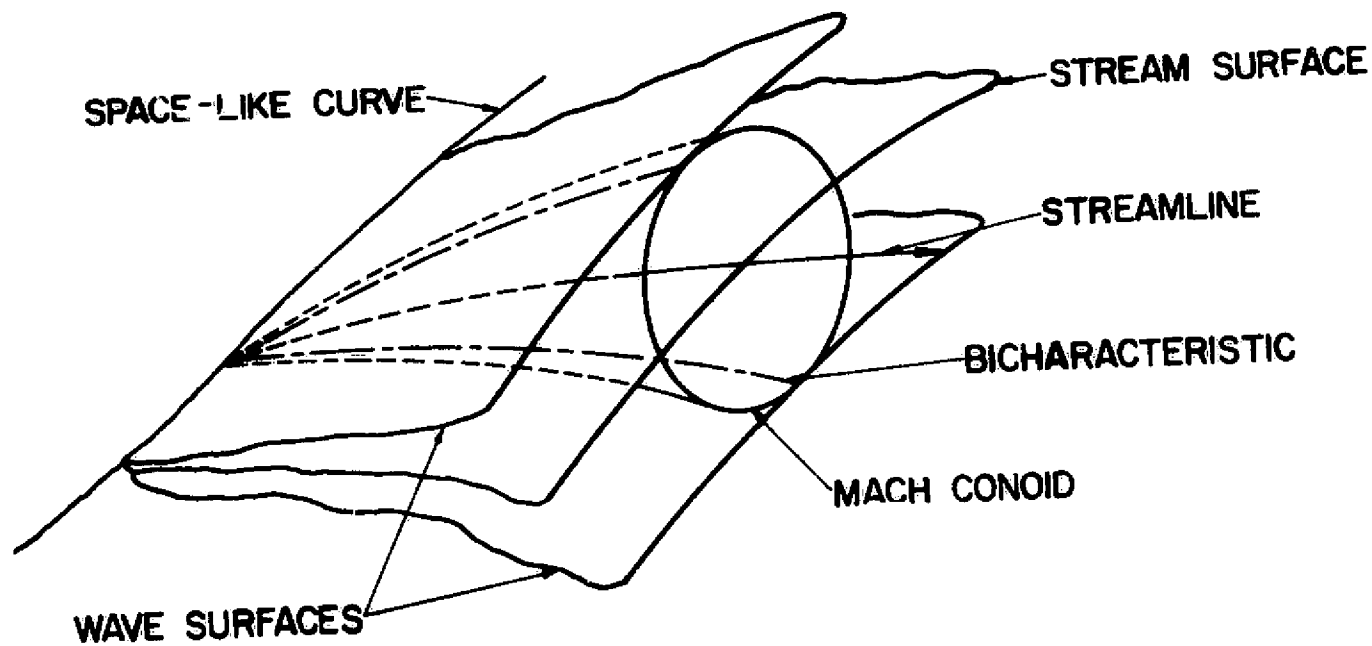


FIGURE B.2. CHARACTERISTIC SURFACES

The coefficient matrix in equation (B.29) is rank two (rank is the number of nonzero rows in the row echelon form of a matrix). The number of independent nontrivial solutions for the ω_i is equal to the order of the coefficient matrix minus its rank, and hence, in this case, is three. From equation (B.29), $\omega_4 = 0$ for all solutions, ω_5 is arbitrary, while $\omega_1, \omega_2, \omega_3$ satisfy the following equation.

$$\omega_1 N_x + \omega_2 N_y + \omega_3 N_z = 0 \quad (\text{B.30})$$

A set of three possible solutions is

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = 0, \quad \omega_5 = 1 \quad (\text{B.31})$$

$$\omega_1 = u, \quad \omega_2 = v, \quad \omega_3 = w, \quad \omega_4 = \omega_5 = 0 \quad (\text{B.32})$$

$$\omega_1 = S_x, \quad \omega_2 = S_y, \quad \omega_3 = S_z, \quad \omega_4 = \omega_5 = 0 \quad (\text{B.33})$$

The vector $\bar{S} = (S_x, S_y, S_z)$ in equation (B.33) lies in the stream surface and is independent of the velocity vector.

On a wave surface, equation (B.21) is valid. That equation may be written as

$$U = a|\bar{N}| \quad (\text{B.34})$$

where $|\bar{N}|$ is the magnitude of the normal to the wave surface. Substituting equation (B.34) into equation (B.16) yields

$$\begin{bmatrix} \rho a |\bar{N}| & 0 & 0 & \rho N_x & 0 \\ 0 & \rho a |\bar{N}| & 0 & \rho N_y & 0 \\ 0 & 0 & \rho a |\bar{N}| & \rho N_z & 0 \\ N_x & N_y & N_z & 0 & a |\bar{N}| \\ 0 & 0 & 0 & a |\bar{N}| & -a^3 |\bar{N}| \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{bmatrix} = 0 \quad (\text{B.35})$$

The coefficient matrix in equation (B.35) is rank four, and, hence, one independent nontrivial solution exists for the ω_i . The solutions for ω_1 , ω_2 , ω_3 , and ω_5 may be expressed in terms of ω_4 . Arbitrarily selecting $\omega_4 = -1$ yields

$$\begin{aligned} \omega_1 &= n_x/a, & \omega_2 &= n_y/a, & \omega_3 &= n_z/a, & \omega_4 &= -1, \\ \omega_5 &= -1/a^2 \end{aligned} \quad (\text{B.36})$$

where $\hat{n} = (n_x, n_y, n_z)$ is the unit normal to the wave surface.

5. COMPATIBILITY RELATIONS

The compatibility relations are obtained by substituting the solutions for the ω_i into equation (B.6). The compatibility relations valid along the stream surfaces are obtained by substituting equations (B.31) to (B.33) into equation (B.6). The results are

$$uP_x + vP_y + wP_z - a^2(u\rho_x + v\rho_y + w\rho_z) = F_e \quad (\text{B.37})$$

$$\begin{aligned} \rho u(uu_x + vu_y + wu_z) + \rho v(uv_x + vv_y + wv_z) + \rho w(uw_x + vw_y \\ + ww_z) + uP_x + vP_y + wP_z = uF_x + vF_y + wF_z \end{aligned} \quad (B.38)$$

$$\begin{aligned} \rho S_x(uu_x + vu_y + wu_z) + \rho S_y(uv_x + vv_y + wv_z) + \rho S_z(uw_x + vw_y \\ + ww_z) + S_x P_x + S_y P_y + S_z P_z = S_x F_x + S_y F_y + S_z F_z \end{aligned} \quad (B.39)$$

Note that equation (B.37) is the same as equation (B.5), which shows that the energy equation is characteristic to begin with.

Equations (B.37) and (B.38) may be written in a form that represents differentiation in the streamline direction only. From equation (B.13), noting that for a streamline $l_x = u$, $l_y = v$, and $l_z = w$, the directional derivative along a streamline is given by

$$\frac{d(\quad)}{dt} = u \frac{\partial(\quad)}{\partial x} + v \frac{\partial(\quad)}{\partial y} + w \frac{\partial(\quad)}{\partial z} \quad (B.40)$$

where t is the time of travel of a fluid particle along the streamline. Using equation (B.40), equations (B.37) and (B.38) may be rewritten as

$$\frac{dP}{dt} - a^2 \frac{d\rho}{dt} = F_e \quad (B.41)$$

$$\rho u \frac{du}{dt} + \rho v \frac{dv}{dt} + \rho w \frac{dw}{dt} + \frac{dP}{dt} = uF_x + vF_y + wF_z \quad (B.42)$$

The compatibility equation that is valid along wave surfaces is obtained by substituting equation (B.36) into equation (B.6). The result is

$$\begin{aligned}
& \rho a n_x (u u_x + v u_y + w u_z) + \rho a n_y (u v_x + v v_y + w v_z) \\
& + \rho a n_z (u w_x + v w_y + w w_z) + (a n_x - u) P_x + (a n_y - v) P_y \\
& + (a n_z - w) P_z - \rho a^2 (u_x + v_y + w_z) = \lambda
\end{aligned} \tag{B.43}$$

where

$$\lambda = a(n_x F_x + n_y F_y + n_z F_z) - F_e \tag{B.44}$$

Equation (B.43) may be written in a form that contains differentiation in the bicharacteristic direction. A bicharacteristic is a ray or generator of the Mach cone. The Mach cone is the reciprocal cone to the cone of normals (see Figure B.1). As a consequence, a bicharacteristic is orthogonal to the surface of the cone of normals. The equation for the cone of normals is given by equation (B.21). Substitution of equation (B.24) into equation (B.21) yields the equation for the surface of the cone of normals in standard form $[f(x,y,z) = \text{constant}]$. Differentiation of this expression to obtain the gradient yields the direction of the bicharacteristic. This gives $\ell_x = (u - a n_x)$, $\ell_y = (v - a n_y)$, and $\ell_z = (w - a n_z)$ in equation (B.13), so that differentiation in the bicharacteristic direction is given by

$$\frac{d(\quad)}{dt} = (u - a n_x) \frac{\partial(\quad)}{\partial x} + (v - a n_y) \frac{\partial(\quad)}{\partial y} + (w - a n_z) \frac{\partial(\quad)}{\partial z} \tag{B.45}$$

In equation (B.45), t is the time of travel of a fluid particle along the streamline that is the axis of the Mach cone. The relationship between the vectors $\bar{\ell}$, $\bar{V}$, and $\hat{n}$ is shown in Figure B.3.

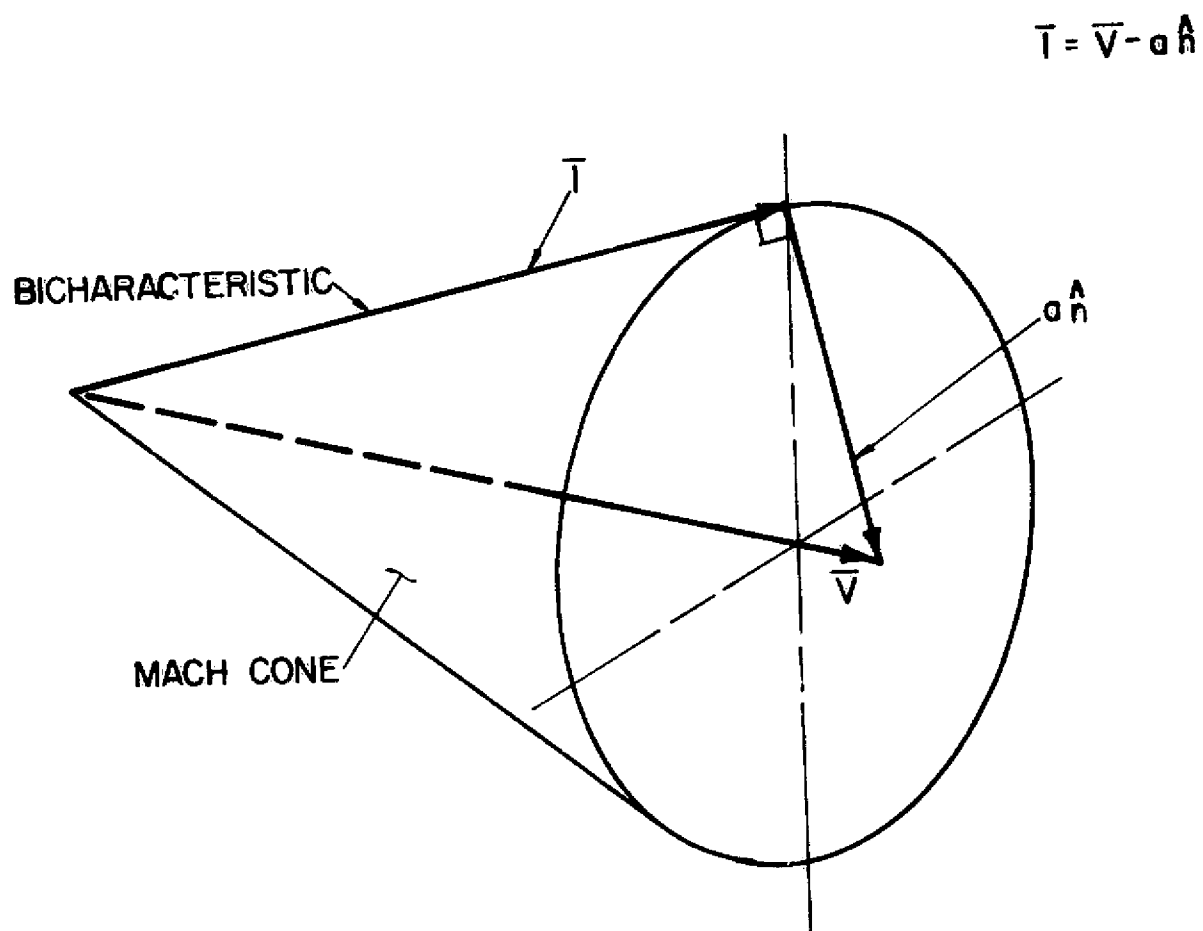


FIGURE B.3. RELATIONSHIP BETWEEN VECTORS $\bar{T}$, $\hat{n}$, and $\bar{V}$

The term

$$\pm \{ \rho a^2 [n_x^2 u_x + n_y^2 v_y + n_z^2 w_z + (u_y + v_x) n_x n_y + (u_z + w_x) n_x n_z + (v_z + w_y) n_y n_z] \}$$

may be added to and subtracted from equation (B.43), and then by employing equation (B.45) the following form of the wave surface compatibility relation may be obtained.

$$\begin{aligned} \rho a n_x \frac{du}{dt} + \rho a n_y \frac{dv}{dt} + \rho a n_z \frac{dw}{dt} - \frac{dP}{dt} = \lambda - \rho a^2 [(n_x^2 - 1) u_x + (n_y^2 - 1) v_y + (n_z^2 - 1) w_z + n_x n_y (u_y + v_x) + n_x n_z (u_z + w_x) + n_y n_z (v_z + w_y)] \end{aligned} \quad (B.46)$$

The terms in brackets in equation (B.46) are known as cross derivatives and represent differentiation in the wave surface in a direction normal to the bicharacteristic direction.

Equations (B.29) and (B.35) determine the number of independent differential compatibility relations valid along a particular stream surface and a particular wave surface, respectively. At any point there exist an infinite number of stream surfaces and wave surfaces. However, the number of independent compatibility relations cannot exceed the number of independent equations of motion. Hence, it is necessary to determine which of the possible combinations of compatibility relations are independent. Ruzanov (32), using a proof in the space of characteristic normals, has shown that for steady three-dimensional isentropic flow two of the stream surface compatibility relations and

the single wave surface compatibility relation applied along three different wave surfaces form an independent set of characteristic equations. Rusanov's results may be extended to the present problem since the principal parts of equations (B.1) to (B.5) are the same as those for isentropic flow. Thus, for the present problem, an independent set of compatibility equations consists of equations (B.41) and (B.42) applied along a streamline, and equation (B.43) [or equation (B.46)] applied along three different wave surfaces.

6. BUTLER'S PARAMETERIZATION OF THE CHARACTERISTIC EQUATIONS

The numerical algorithm that is employed in the present investigation is based on a second-order scheme devised by D.S. Butler (24). This scheme has been used by Ransom, Hoffman, and Thompson (9) to compute isentropic steady three-dimensional nozzle flows, and by Cline and Hoffman (25) to compute chemically-reacting steady three-dimensional nozzle flows.

In this section, Butler's parameterization of the characteristic equations is presented. The discussion below is limited to the particular application of Butler's method to the present problem. An excellent review of Butler's general method is given in Ransom, Hoffman, and Thompson (9).

For Butler's scheme to be applicable, the characteristic determinant must be composed of a quadratic factor and a repeated linear factor. The determinant of the coefficient matrix in equation (B.16) is the characteristic determinant for the present problem, and by examination of equation (B.18) it is seen that it is composed of the required factors. The quadratic factor corresponds to the wave surfaces. The

envelope of all wave surfaces at a point is the Mach cone. The line of tangency between a particular wave surface and the Mach cone is a bi-characteristic. The linear factor corresponds to the stream surfaces. The axis of the envelope of all stream surfaces at a point is a streamline. Butler's method assumes that for the linear factor, differentiation can be expressed solely along the axis of the envelope of the corresponding characteristic surfaces. Examination of equations (B.41) and (B.42) demonstrates that this condition is applicable.

As discussed in the first section of this appendix, if the system of governing partial differential equations has differentiation occurring in n -space, then differentiation in the characteristic surfaces occurs in $(n-1)$ -space (i.e., differentiation is performed in a manifold of one lower dimension). As a result, for three-dimensional flow ($n=3$), the general form of a compatibility relation valid along a characteristic surface may be written as

$$E_v(\partial u_v / \partial x_i^1) + F_v(\partial u_v / \partial x_2^1) = D \quad (B.47)$$

where the repeated index v implies summation over the range of 1 to 5, x_i^1 ($i=1,2$) denotes two independent directions in the characteristic surface, u_v ($v=1$ to 5) denotes the dependent variables, and E_v , F_v ($v=1$ to 5), and D are general functions of x_i^1 and u_v . For stream surfaces, differentiation may be expressed solely in the streamline direction [see equations (B.41) and (B.42)]. Consequently, in the following, the discussion will be limited to the wave surfaces.

For steady three-dimensional flow, Butler introduced the following parametric representation for a bicharacteristic.

$$dx_i = (u_i + c\alpha_i \cos\theta + c\beta_i \sin\theta)dt \quad (i=1,2,3) \quad (B.48)$$

In equation (B.48), x_i ($i=1,2,3$) denotes the three cartesian coordinates x , y , and z , respectively, u_i ($i=1,2,3$) denotes the corresponding velocity components u , v , and w , respectively, θ is a parametric angle denoting a particular element of the Mach cone and has the range $0 \leq \theta \leq 2\pi$, t is the time of travel of a fluid particle along the streamline that is the axis of the Mach cone, and c is defined by

$$c^2 = a^2 q^2 / (q^2 - a^2) \quad (B.49)$$

where q is the velocity magnitude and a is the sonic speed. The vectors α_i and β_i are parametric unit vectors with α_i , β_i , and u_i/q ($i=1,2,3$) forming an orthonormal set. A geometrical representation of this parameterization is given in Figure B.4.

The direction specified by equation (B.48) lies in the wave surface and is in the bicharacteristic direction. A direction in the wave surface and orthogonal to the bicharacteristic direction may be written in parametric form as

$$m_i = c\beta_i \cos\theta - c\alpha_i \sin\theta \quad (i=1,2,3) \quad (B.50)$$

Verification of the orthogonality of the directions given by equations (B.48) and (B.50) may be accomplished by forming the dot product $(m_i dx_i)$ and using the orthonormality relations

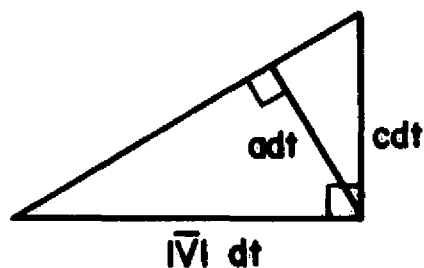
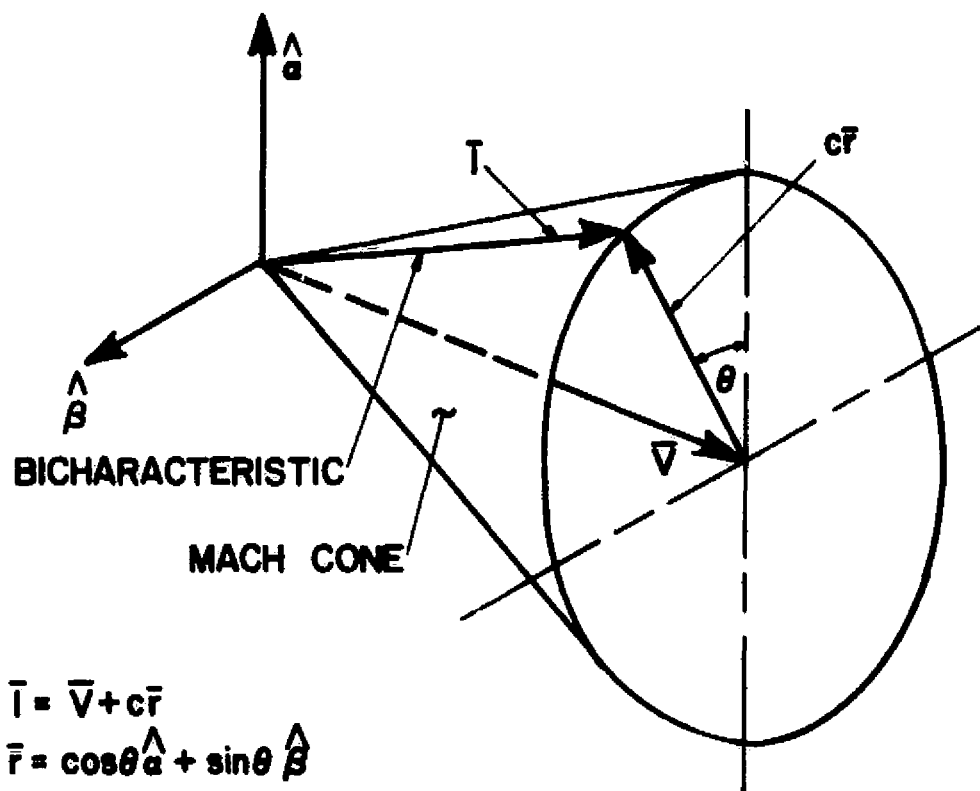


FIGURE B.4. BICHARACTERISTIC PARAMETERIZATION

$$u_i \alpha_i = u_i \beta_i = \alpha_i \beta_i = 0 \quad (B.51)$$

$$\alpha_i \alpha_i = \beta_i \beta_i = u_i u_i / q^2 = 1$$

By considering equation (B.47) and selecting x'_1 and x'_2 as the directions given by equations (B.48) and (B.50), respectively, the following form of the wave surface compatibility relation is obtained.

$$A_v (u_i + c \alpha_i \cos \theta + c \beta_i \sin \theta) \frac{\partial u_v}{\partial x_i} = B + C_v (c \beta_i \cos \theta - c \alpha_i \sin \theta) \frac{\partial u_v}{\partial x_i} \quad (B.52)$$

In equation (B.52), A_v , B , and C_v are functions of θ , u_v , and x_i . Employing equation (B.13), and noting from equation (B.48) that along a bicharacteristic

$$l_i = u_i + c \alpha_i \cos \theta + c \beta_i \sin \theta \quad (i=1,2,3) \quad (B.53)$$

equation (B.52) may be written as

$$A_v \frac{du}{dl} = B + C_v (c \beta_i \cos \theta - c \alpha_i \sin \theta) \frac{\partial u_v}{\partial x_i} \quad (B.54)$$

where the operator $d(.)/dl$ represents the directional derivative along the bicharacteristic. The general forms of the coefficients A_v , B , and C_v are given by Butler as

$$A_v = A_{1v} + A_{2v} \cos \theta + A_{3v} \sin \theta \quad (B.55)$$

$$B = B_1 + B_2 \cos \theta + B_3 \sin \theta \quad (B.56)$$

$$C_v = C_{1v} + C_{2v} \cos \theta + C_{3v} \sin \theta \quad (B.57)$$

where the A_{kv} , B_k , and C_{kv} ($k=1,2,3$ and $v=1$ to 5) are independent of θ .

In addition to the parametric wave surface compatibility relation, given by equation (B.54), Butler also developed a noncharacteristic relation which is applied along a streamline. This noncharacteristic relation is used in the numerical scheme in conjunction with the wave surface compatibility relation applied along four different bicharacteristics, and permits the formulation of three independent linear combinations of these five equations which do not contain cross derivatives at the solution point. The cross derivative terms [see equation (B.46)] represent differentiation in the wave surface but in a direction orthogonal to the bicharacteristic direction [i.e., differentiation in the direction given by equation (B.50)]. Butler presents the noncharacteristic relation in the form

$$A_{1v} \frac{du_v}{d\lambda} = B_1 + (C_{2v} c\beta_i - C_{3v} c\alpha_i) \frac{\partial u_v}{\partial x_i} \quad (B.58)$$

where the operator $d()/d\lambda$ represents the directional derivative along the streamline. The coefficients A_{1v} , B_1 , C_{2v} , and C_{3v} ($v=1$ to 5) in equation (B.58) are obtained by inspecting the form of equation (B.54) and then using equations (B.55), (B.56), and (B.57).

For the present problem, the actual form of the parametric wave surface compatibility relation, equation (B.54), may be obtained by substituting the appropriate parametric form of the wave surface unit normal into the compatibility relation, equation (B.43). The normal to the wave surface is also the normal to the Mach cone at a point common to both surfaces. The quadric surface of the Mach cone is

represented by equation (B.27), repeated below.

$$[u_i u_j - (q^2 - a^2) \delta_{ij}] dx_i dx_j = 0 \quad (B.27)$$

Substituting the parametric form for dx_j , given by equation (B.48), into equation (B.27) yields

$$[u_i u_j - (q^2 - a^2) \delta_{ij}] (u_j + c \alpha_j \cos \theta + c \beta_j \sin \theta) dx_i = 0 \quad (B.59)$$

The i th component of the normal N_i to this surface is

$$N_i = [u_i u_j - (q^2 - a^2) \delta_{ij}] (u_j + c \alpha_j \cos \theta + c \beta_j \sin \theta) \quad (i=1,2,3) \quad (B.60)$$

Employing the orthonormality conditions given by equation (B.51), equation (B.60) may be written as

$$N_i = a^2 [u_i - (q^2/c) (\alpha_i \cos \theta + \beta_i \sin \theta)] \quad (i=1,2,3) \quad (B.61)$$

Dividing equation (B.61) by the magnitude of the normal $[|\mathbf{N}| = (N_i N_i)^{1/2}]$ and using equation (B.51), the parametric form of the wave surface unit normal is obtained.

$$n_i = (a/c) (c u_i / q^2 - \alpha_i \cos \theta - \beta_i \sin \theta) \quad (i=1,2,3) \quad (B.62)$$

Substituting equation (B.62) and the orthonormality relation

$$\alpha_i \alpha_j + \beta_i \beta_j + u_i u_j / q^2 = \delta_{ij} \quad (B.63)$$

into the wave surface compatibility relation, equation (B.43), gives the following parametric form of that equation

$$\begin{aligned} \frac{dP}{dt} + \rho c(\alpha_i \cos\theta + \beta_i \sin\theta) \frac{du_i}{dt} = \Phi - \rho c^2(\alpha_i \sin\theta \\ - \beta_i \cos\theta)(\alpha_j \sin\theta - \beta_j \cos\theta) \frac{\partial u_i}{\partial x_j} \end{aligned} \quad (B.64)$$

where

$$\Phi = - (c^2/a^2)\lambda \quad (B.65)$$

The operator $d(\)/dt$ in equation (B.64) denotes differentiation in the bicharacteristic direction.

It should be noted that the directional derivatives in equations (B.46) and (B.64) are not identical. The directional derivative in equation (B.46) is based on equation (B.45). Substitution of the parametric unit normal, given by equation (B.62), into equation (B.45) yields

$$\frac{d(\)}{dt} = (a^2/c^2)(u_i + c\alpha_i \cos\theta + c\beta_i \sin\theta) \frac{\partial(\)}{\partial x_i} \quad (B.66)$$

The directional derivative in equation (B.64) is given by

$$\frac{d(\)}{dt} = (u_i + c\alpha_i \cos\theta + c\beta_i \sin\theta) \frac{\partial(\)}{\partial x_i} \quad (B.67)$$

Hence, the two expressions differ by the factor (a^2/c^2) .

Finally, it remains to determine the actual form of the noncharacteristic relation, equation (B.58). Denote u_v ($v=1$ to 5) and x_i ($i=1,2,3$) in equations (B.54) and (B.58) by

$$\begin{aligned} u_1 = u, \quad u_2 = v, \quad u_3 = w, \quad u_4 = P, \quad u_5 = \rho \\ x_1 = x, \quad x_2 = y, \quad x_3 = z \end{aligned} \quad (B.68)$$

By inspection of equation (B.64), and use of equations (B.68), (B.55), (B.56), and (B.57), the noncharacteristic relation is seen to be

$$\frac{dP}{dt} = \sigma - \rho c^2 (\alpha_i \alpha_j + \beta_i \beta_j) \frac{\partial u_i}{\partial x_j} \quad (B.69)$$

where

$$\sigma = (c^2/a^2)F_e - (c^2/q^2)(uF_x + vF_y + wF_z) \quad (B.70)$$

The operator $d(\)/dt$ in equation (B.69) denotes the directional derivative along a streamline.

In summary, Butler has developed a bicharacteristic parameterization given by equation (B.48). The corresponding parametric form of the wave surface compatibility relation is given by equation (B.64). Butler also developed a noncharacteristic relation, given by equation (B.69), which is applied along a streamline. These relations, along with the stream surface compatibility relations, equations (B.41) and (B.42), constitute the system of compatibility relations. The use of this system of equations in the various unit processes is presented in Appendix E.

APPENDIX C

INTERPOLATION

1. INTRODUCTION

In the course of computing the flow field, a number of situations arise which require interpolation. To this end, univariate, bivariate, and trivariate interpolation polynomials are employed in the numerical algorithm. These interpolation schemes are presented in this appendix.

2. UNIVARIATE INTERPOLATION

Univariate interpolation is required in geometry description, calculation of the transport forcing terms, and in determination of the properties along a space curve formed by the locus of shock wave solution points. Applications to geometry description and transport term computation are discussed in Appendices D and G, respectively. The application to the determination of properties along a shock wave is discussed here.

When a shock wave intersects either a solid boundary or a solution plane (a plane of constant x), a space curve is defined as illustrated in Figure C.1. Interpolated values of position, shock wave angle, and flow properties are required along this curve. For this purpose, the quadratic polynomial

$$f(\theta) = a_1 + a_2\theta + a_3\theta^2 \quad (C.1)$$

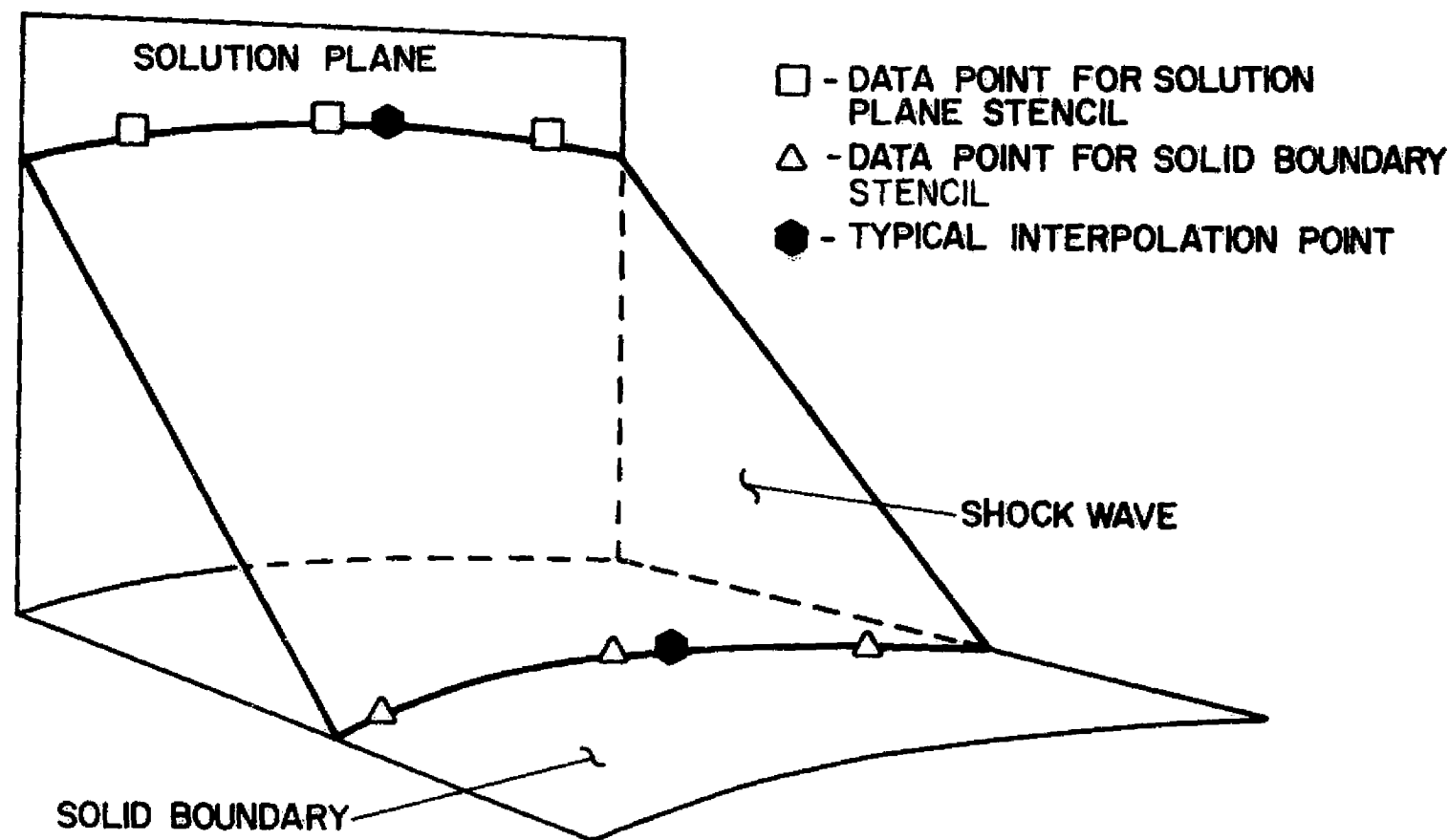


FIGURE C.1. POINT STENCILS FOR UNIVARIATE INTERPOLATION

is employed, where $f(\theta)$ denotes a general function expressed in terms of the polar angle θ given by

$$\theta = \tan^{-1}(z/y) \quad (C.2)$$

where y and z are the coordinates of a point on the space curve. The coefficients a_i ($i=1,2,3$) in equation (C.1) are determined by fitting this expression to three data points on the space curve, and, as a consequence, a system of three simultaneous linear equations must be solved for the coefficients a_i of each function representation. The solution to this system of equations is obtained using a Gaussian elimination method with complete pivoting (40).

Figure C.1 illustrates typical data point stencils used for determining coefficients in equation (C.1). The fit point array consists of a base point, which is the point closest to the position of the interpolated point, and the immediate neighbors of the base point.

3. BIVARIATE INTERPOLATION

Bivariate interpolation is required for property determination in a given solution plane (a plane of constant x). Two types of bivariate interpolation polynomials are employed in the numerical algorithm. They are a linear bivariate polynomial whose three coefficients are determined by fitting this expression to three data points, and a quadratic bivariate polynomial whose six coefficients are determined by a least squares fitting of nine data points.

The linear bivariate polynomial is used in the single application when a streamline-shock wave intersection point is sufficiently close to the current solution plane so that an interior point unit

process on the downstream side of the shock wave is not performed. In that case the projection of the streamline onto the solution plane and subsequent property interpolation in this plane is performed. The bivariate interpolation polynomial used in this case is

$$f(y,z) = a_1 + a_2y + a_3z \quad (C.3)$$

where $f(y,z)$ denotes a general function of the coordinates y and z . The coefficients a_i ($i=1,2,3$) in equation (C.3) are determined by fitting this expression to three data points. This yields a system of three simultaneous linear equations for the coefficients a_i of each function representation. This system of equations is solved using a Gaussian elimination method with complete pivoting [as was done for equation (C.1)].

A typical data point stencil used for determining the coefficients in equation (C.3) is illustrated in Figure C.2. Two shock wave solution points and a field point constitute the fit point array.

In all other situations which require bivariate interpolation, the quadratic polynomial

$$f(y,z) = a_1 + a_2y + a_3z + a_4yz + a_5y^2 + a_6z^2 \quad (C.4)$$

is employed, where $f(y,z)$ is a general function of the coordinates y and z . The coefficients a_i ($i=1$ to 6) in equation (C.4) are determined by a least squares fit of nine points. Using the standard theory of least squares (40), the system of normal equations which determines the coefficients in equation (C.4) is

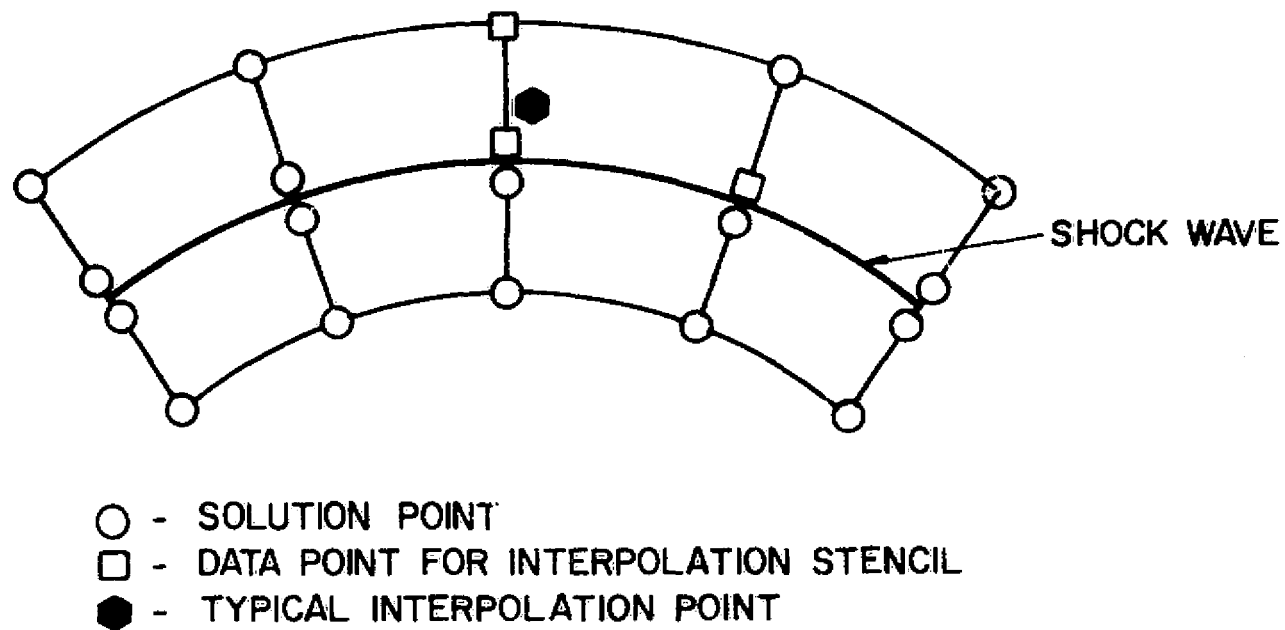


FIGURE C.2. POINT STENCIL FOR LINEAR BIVARIATE INTERPOLATION

$$9a_1 + \sum y_i a_2 + \sum z_i a_3 + \sum y_i z_i a_4 + \sum y_i^2 a_5 + \sum z_i^2 a_6 = \sum f_i \quad (C.5)$$

$$\begin{aligned} & \sum y_i a_1 + \sum y_i^2 a_2 + \sum y_i z_i a_3 + \sum y_i^2 z_i a_4 + \sum y_i^3 a_5 \\ & + \sum y_i z_i^2 a_6 = \sum y_i f_i \end{aligned} \quad (C.6)$$

$$\begin{aligned} & \sum z_i a_1 + \sum y_i z_i a_2 + \sum z_i^2 a_3 + \sum y_i z_i^2 a_4 + \sum y_i^2 z_i a_5 \\ & + \sum z_i^3 a_6 = \sum z_i f_i \end{aligned} \quad (C.7)$$

$$\begin{aligned} & \sum y_i z_i a_1 + \sum y_i^2 z_i a_2 + \sum y_i z_i^2 a_3 + \sum y_i^2 z_i^2 a_4 + \sum y_i^3 z_i a_5 \\ & + \sum y_i z_i^3 a_6 = \sum y_i z_i f_i \end{aligned} \quad (C.8)$$

$$\begin{aligned} & \sum y_i^2 a_1 + \sum y_i^3 a_2 + \sum y_i^2 z_i a_3 + \sum y_i^3 z_i a_4 + \sum y_i^4 a_5 \\ & + \sum y_i^2 z_i^2 a_6 = \sum y_i^2 f_i \end{aligned} \quad (C.9)$$

$$\begin{aligned} & \sum z_i^2 a_1 + \sum y_i z_i^2 a_2 + \sum z_i^3 a_3 + \sum y_i z_i^3 a_4 + \sum y_i^2 z_i^2 a_5 \\ & + \sum z_i^4 a_6 = \sum z_i^2 f_i \end{aligned} \quad (C.10)$$

In equations (C.5) to (C.10), the $\sum$ sign implies summation over the range of 1 to 9, while the subscript i denotes the i th data point ($i=1$ to 9). This system of simultaneous linear equations has a symmetric coefficient matrix and is solved using a Gaussian elimination method with pivoting in the main diagonal.

Figure C.3 illustrates typical data point stencils used in determining the coefficients in equation (C.4). Basically, there are two

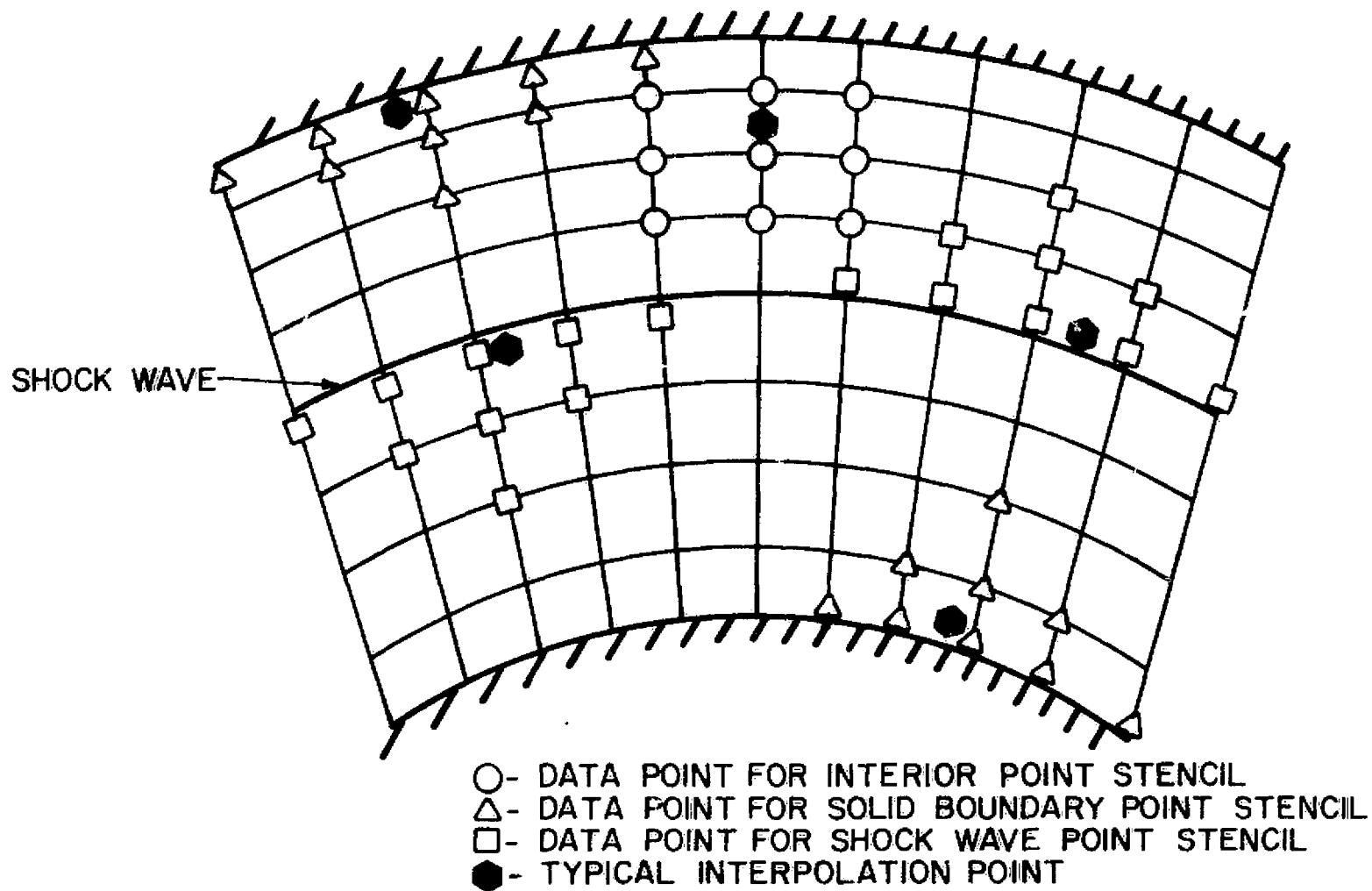


FIGURE C . 3.
POINT STENCILS FOR QUADRATIC BIVARIATE INTERPOLATION

types of stencils: interior point and boundary point. Since the shock wave mathematically represents a discontinuity, the boundary point stencil must be employed when the interpolation base point (the data point closest to the interpolated point) is on the shock wave. The fit point array consists of the base point and its eight immediate neighbors. Special logic in the computer program is used to insure that no stencil bridges the shock wave.

4. TRIVARIATE INTERPOLATION

Trivariate interpolation is required for property determination on the surface of a solid boundary (a stream surface) and for property determination on the upstream and downstream sides of the shock wave. Two types of trivariate interpolation polynomials are employed in the numerical algorithm. They are a linear trivariate polynomial whose four coefficients are determined by fitting this expression to four data points, and a quadratic trivariate polynomial whose eight coefficients are determined by a least squares fitting of fourteen data points.

The linear trivariate polynomial is used in the single application for property determination on the upstream side of the shock wave surface. This polynomial has the form

$$f(x,y,z) = a_1 + a_2x + a_3y + a_4z \quad (C.11)$$

where $f(x,y,z)$ is a general function of the coordinates x , y , and z . The coefficients a_i ($i=1,2,3,4$) in equation (C.11) are determined by fitting this expression to four data points. Hence, a system of four simultaneous linear equations must be solved for the coefficient a_i

of each function representation. This system of equations is solved using a Gaussian elimination method with complete pivoting [as was done for equations (C.1) and (C.3)].

A typical data point stencil used for determining the coefficients in equation (C.11) is illustrated in Figure C.4. Three data points are located on one space curve and one data point is located on the other space curve.

In all other situations which require trivariate interpolation, the quadratic polynomial

$$f(x,y,z) = a_1 + a_2y + a_3z + a_4yz + a_5y^2 + a_6z^2 + a_7xy + a_8xz \quad (C.12)$$

is employed, where $f(x,y,z)$ is a general function dependent on the coordinates x , y , and z . The coefficients a_i ($i=1$ to 8) in equation (C.12) are determined by a least squares fit of fourteen data points. From the theory of least squares, the system of normal equations which determines the coefficients in equation (C.12) is

$$14a_1 + \sum y_i a_2 + \sum z_i a_3 + \sum y_i z_i a_4 + \sum y_i^2 a_5 + \sum z_i^2 a_6 + \sum x_i y_i a_7 + \sum x_i z_i a_8 = \sum f_i \quad (C.13)$$

$$\sum y_i a_1 + \sum y_i^2 a_2 + \sum y_i z_i a_3 + \sum y_i^2 z_i a_4 + \sum y_i^3 a_5 + \sum y_i z_i^2 a_6 + \sum x_i y_i^2 a_7 + \sum x_i y_i z_i a_8 = \sum y_i f_i \quad (C.14)$$

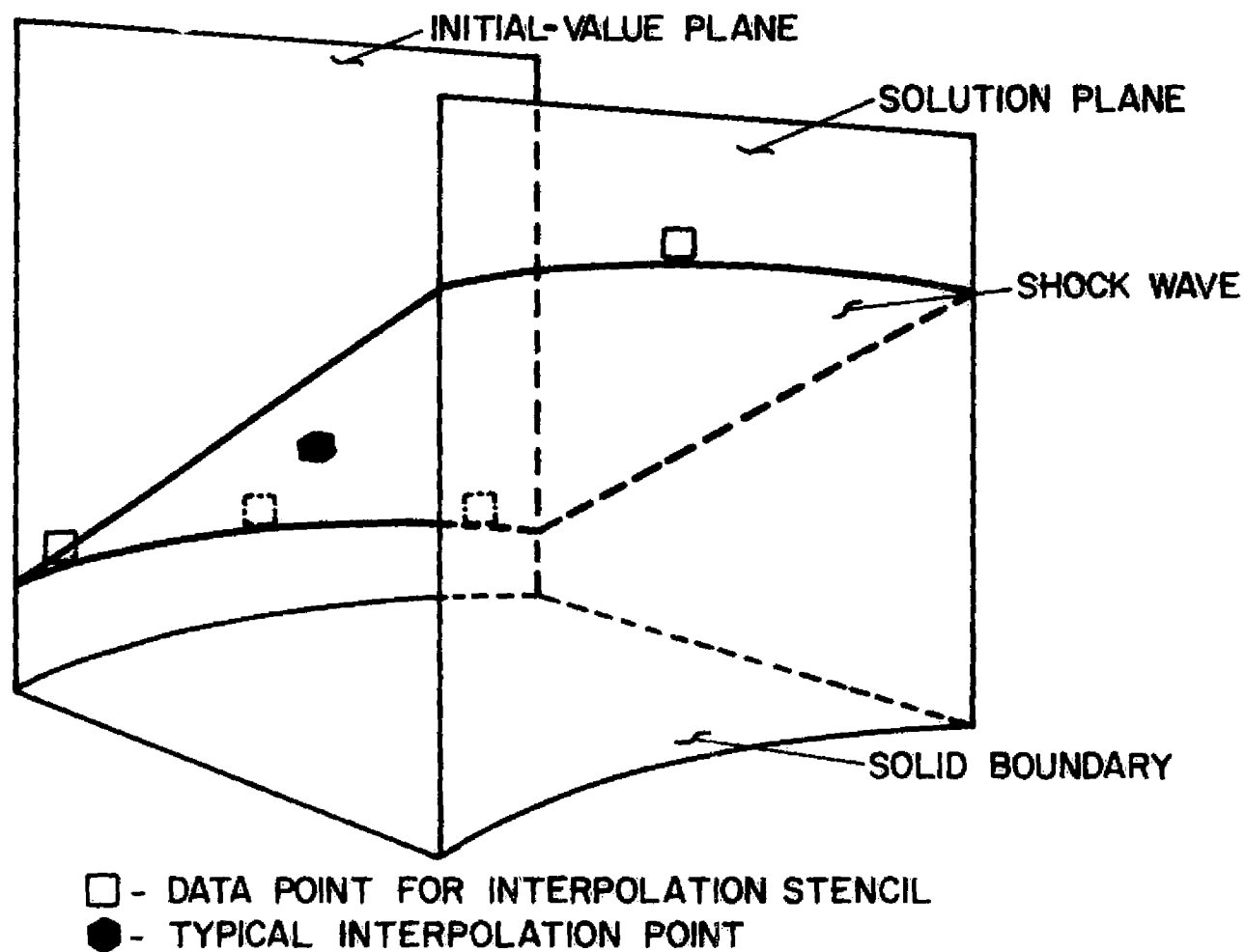


FIGURE C.4. POINT STENCIL FOR LINEAR TRIVARIATE INTERPOLATION

$$\begin{aligned} \sum z_i a_1 + \sum y_i z_i a_2 + \sum z_i^2 a_3 + \sum y_i z_i^2 a_4 + \sum y_i^2 z_i a_5 + \sum z_i^3 a_6 \\ + \sum x_i y_i z_i a_7 + \sum x_i z_i^2 a_8 = \sum z_i f_i \end{aligned} \quad (C.15)$$

$$\begin{aligned} \sum y_i z_i a_1 + \sum y_i^2 z_i a_2 + \sum y_i z_i^2 a_3 + \sum y_i^2 z_i^2 a_4 + \sum y_i^3 z_i a_5 \\ + \sum y_i z_i^3 a_6 + \sum x_i y_i^2 z_i a_7 + \sum x_i y_i z_i^2 a_8 = \sum y_i z_i f_i \end{aligned} \quad (C.16)$$

$$\begin{aligned} \sum y_i^2 a_1 + \sum y_i^3 a_2 + \sum y_i^2 z_i a_3 + \sum y_i^3 z_i a_4 + \sum y_i^4 a_5 + \sum y_i^2 z_i^2 a_6 \\ + \sum x_i y_i^3 a_7 + \sum x_i y_i^2 z_i a_8 = \sum y_i^2 f_i \end{aligned} \quad (C.17)$$

$$\begin{aligned} \sum z_i^2 a_1 + \sum y_i z_i^2 a_2 + \sum z_i^3 a_3 + \sum y_i z_i^3 a_4 + \sum y_i^2 z_i^2 a_5 + \sum z_i^4 a_6 \\ + \sum x_i y_i z_i^2 a_7 + \sum x_i z_i^3 a_8 = \sum z_i^2 f_i \end{aligned} \quad (C.18)$$

$$\begin{aligned} \sum x_i y_i a_1 + \sum x_i y_i^2 a_2 + \sum x_i y_i z_i a_3 + \sum x_i y_i^2 z_i a_4 + \sum x_i y_i^3 a_5 \\ + \sum x_i y_i z_i^2 a_6 + \sum x_i^2 y_i^2 a_7 + \sum x_i^2 y_i z_i a_8 = \sum x_i y_i f_i \end{aligned} \quad (C.19)$$

$$\begin{aligned} \sum x_i z_i a_1 + \sum x_i y_i z_i a_2 + \sum x_i z_i^2 a_3 + \sum x_i y_i z_i^2 a_4 + \sum x_i y_i^2 z_i a_5 \\ + \sum x_i z_i^3 a_6 + \sum x_i^2 y_i z_i a_7 + \sum x_i^2 z_i^2 a_8 = \sum x_i z_i f_i \end{aligned} \quad (C.20)$$

In equations (C.13) to (C.20), the $\sum$ sign implies summation over the range of 1 to 14, while the subscript i denotes the i th data point ($i=1$ to 14). This system of simultaneous linear equations has a symmetric coefficient matrix and is solved using a Gaussian elimination method with pivoting in the main diagonal [as was done for equation (C.4)].

Figure C.5 illustrates typical data point stencils used in determining the coefficients in equation (C.12). The fit point array consists of seven data points along each of the appropriate space curves on either the shock wave or the solid boundary.

It should be noted that a ten term quadratic trivariate polynomial, with coefficients determined by a least squares fit of fourteen data points, was tried in place of equation (C.12). Use of this polynomial in flows with axial symmetry, however, did not produce results which were as symmetrical (especially for transverse velocity components) as those produced by equation (C.12). This result could possibly be due to the effects of ill-conditioning as is discussed in Hamming (40). Furthermore, scaling of the dependent variables did not appear to produce any improvement.

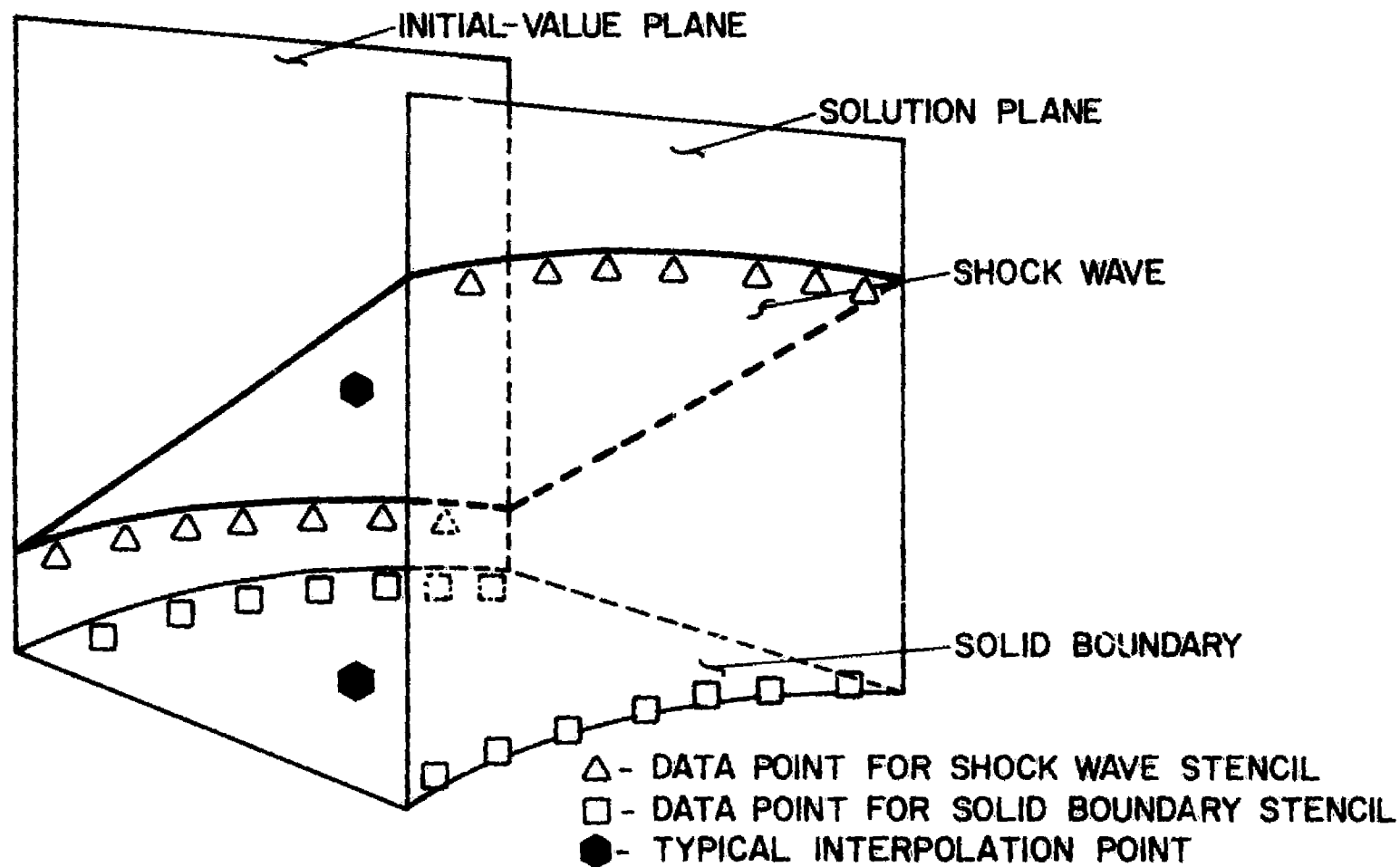


FIGURE C.5. POINT STENCILS FOR QUADRATIC TRIVARIATE INTERPOLATION

APPENDIX D
SURFACE REPRESENTATIONS, AND STREAMLINE- AND
BICHARACTERISTIC-SURFACE INTERSECTIONS

1. INTRODUCTION

The procedures employed for representing the solid boundary and shock wave surfaces are presented in this appendix. The technique used for determining the intersection point of either a streamline with the shock wave, or a bicharacteristic with either the shock wave or the solid boundary, is also discussed.

2. SOLID BOUNDARY SURFACES

The centerbody and cowl surfaces are specified in the computer program by a separate geometry module that has the capability to describe a variety of axisymmetric contours. More arbitrary geometries, such as those having elliptical or superelliptical cross sections, may be considered by supplying an appropriate replacement module. In general, to specify a surface completely, its functional form $[f(x,y,z) = \text{constant}]$ and its gradient at any point $[\nabla f(x,y,z)]$ must be available.

The existing geometry module, which describes axisymmetric contours, divides the axial (x) domain into a number of intervals. In any interval, the body radius may be specified by either tabular input, or by supplying the coefficients in a cubic polynomial written as a

function of x . For the tabular input case, linear interpolation is performed to obtain the radius $r(x)$ between the points (x_i, r_i) and (x_{i+1}, r_{i+1}) where $(x_i \leq x < x_{i+1})$. Alternatively, employing the cubic polynomial

$$r(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$$(x_i \leq x < x_{i+1}) \quad (D.1)$$

requires that the coefficients a_i , b_i , c_i , and d_i be supplied for the i th interval (these coefficients must be externally generated). Since equation (D.1) is a cubic, slope and curvature can be matched at the junction point between two adjacent intervals (i.e., spline fits can be employed).

3. SHOCK WAVE SURFACE

Some of the unit processes, which are described in Appendix E, require an analytical representation for the shock wave surface. During the course of the program development, a number of different representations were devised, including the fitting of both planar surfaces and quadric surfaces to locally approximate the shock wave surface. The quadric surface formulation displayed a tendency to produce a (local) surface with undulations. The planar surface representation did not exhibit this effect, and, for fine mesh spacings, produced results essentially the same as the representation that was ultimately selected for use in the numerical algorithm. However, the accuracy of the planar surface representation suffered at coarse mesh spacings. The shock wave surface formulation that was selected for use in the algorithm is presented below.

The shock wave surface is represented as a family of straight lines between two space curves, as illustrated in Figure D.1. The space curves represent either the intersection of the shock wave with a solution plane (which is a plane of constant x), or the intersection of the shock wave with a solid boundary (i.e., an interplanar ring of shock wave solution points). Each space curve is represented by the two quadratic expressions

$$r_i(\theta) = a_i + b_i\theta + c_i\theta^2 \quad (i=1,2) \quad (D.2)$$

$$x_i(\theta) = d_i + e_i\theta + f_i\theta^2 \quad (i=1,2) \quad (D.3)$$

where r_i is the radius of a point on space curve i ($i=1,2$), x_i is the corresponding axial position of a point on space curve i , and θ is the polar angle given by

$$\theta = \tan^{-1}(z/y) \quad (D.4)$$

where y and z are the coordinates of a point on the space curve. In equations (D.2) and (D.3), the coefficients a_i to f_i ($i=1,2$) are determined by fitting these expressions to three known points on each space curve as described in Appendix C. When the space curve lies in a solution plane, x of course has no θ dependency.

Once equations (D.2) and (D.3) are determined for the two space curves, the shock wave surface is represented as an infinite family of straight lines between the two space curves, where each straight line falls in a meridional plane (i.e., a plane of constant θ). Consequently, for a given value of θ and x , the shock wave surface is represented by the linear interpolation formula

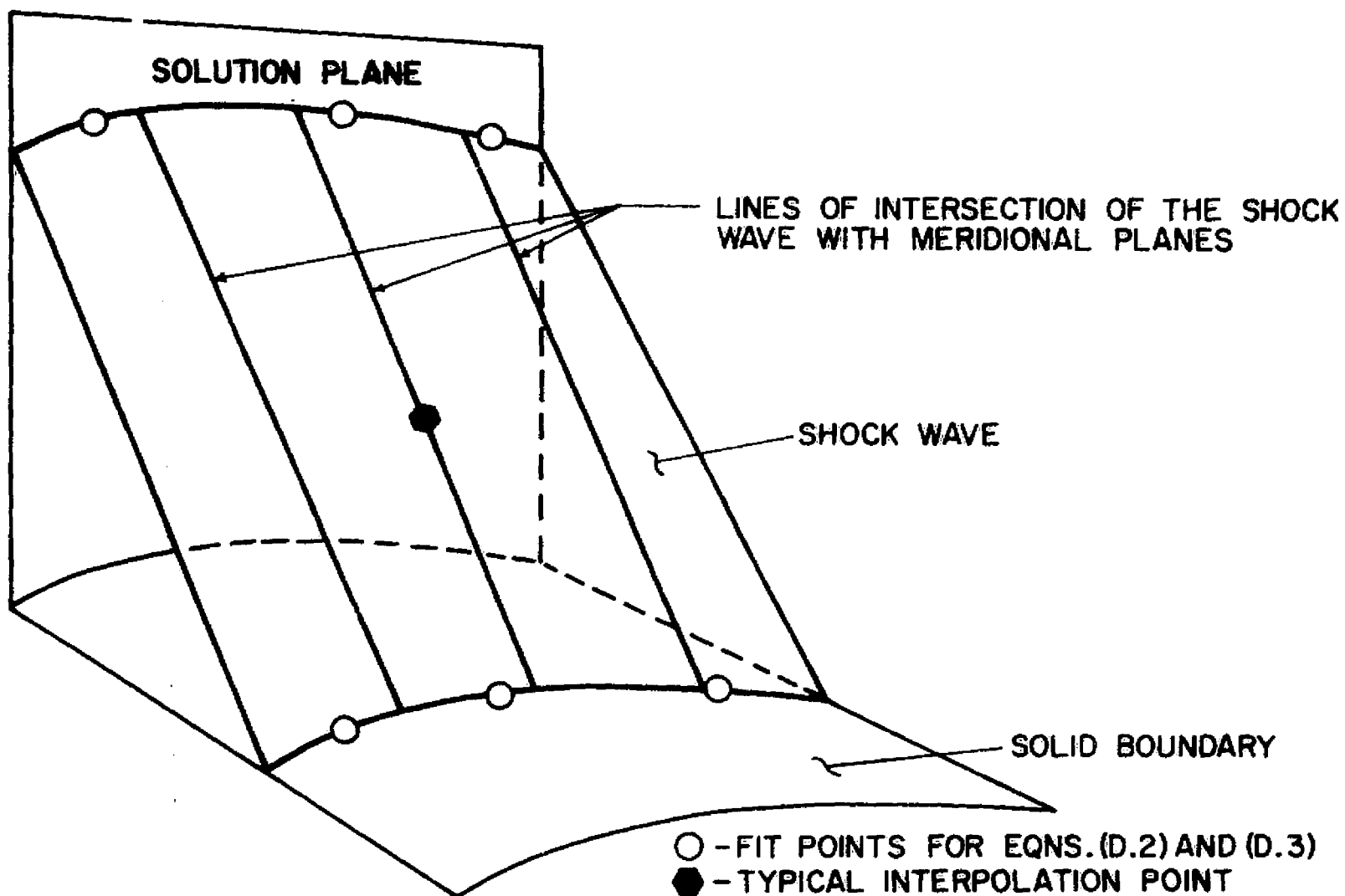


FIGURE D.1. SHOCK WAVE SURFACE REPRESENTATION

$$r(x,\theta) = \frac{(x - x_2(\theta))}{(x_1(\theta) - x_2(\theta))} r_1(\theta) + \frac{(x - x_1(\theta))}{(x_2(\theta) - x_1(\theta))} r_2(\theta) \quad (D.5)$$

In equation (D.5), $r(x,\theta)$ is the shock wave radius at axial position x and polar angle θ , $r_1(\theta)$ and $x_1(\theta)$ are given by equations (D.2) and (D.3), respectively, for one of the space curves, and $r_2(\theta)$ and $x_2(\theta)$ are given by equations (D.2) and (D.3), respectively, for the other space curve (see Figure D.1). A strong point of this representation is that a smooth (local) surface is produced because linear interpolation is performed for the shock wave radius in a meridional plane, while transverse curvature information is introduced through equations (D.2) and (D.3).

4. STREAMLINE- AND BICHARACTERISTIC-SURFACE INTERSECTIONS

A number of unit processes require determining the intersection point of either a streamline with the shock wave, or a bicharacteristic with either the shock wave or a solid boundary. The technique used is the same for all cases and is presented below.

A streamline or bicharacteristic may be represented by the equation

$$dx_i = \Gamma_i dt \quad (i=1,2,3) \quad (D.6)$$

where x_i ($i=1,2,3$) denotes the three cartesian coordinates x , y , and z , respectively, and t is a parameter proportional to the length of the streamline or bicharacteristic. For a streamline, the parameter Γ_i in equation (D.6) is given by

$$\Gamma_i = u_i \quad (i=1,2,3) \quad (D.7)$$

where u_i ($i=1,2,3$) denotes the velocity components u , v , and w , respectively. For a bicharacteristic, Γ_i is given by

$$\Gamma_i = u_i + c\alpha_i \cos\phi + c\beta_i \sin\phi \quad (i=1,2,3) \quad (D.8)$$

where α_i , β_i , ϕ , and c are the parameters employed in Butler's parameterization of the Mach cone (24), which is discussed in Appendix B.

Using equation (D.6), the following equation may be written.

$$dx/\Gamma_1 = dy/\Gamma_2 = dz/\Gamma_3 \quad (D.9)$$

Solving equation (D.9) simultaneously, the linear expressions

$$y = [y_k - (\Gamma_2/\Gamma_1)x_k] + (\Gamma_2/\Gamma_1)x \quad (D.10)$$

$$z = [z_k - (\Gamma_3/\Gamma_1)x_k] + (\Gamma_3/\Gamma_1)x \quad (D.11)$$

may be obtained, where x_k , y_k , and z_k are the coordinates of a known point on the streamline or bicharacteristic, while x , y , and z represent the coordinates of the point of intersection of the streamline or bicharacteristic with a surface (see Figure D.2).

An iterative procedure is employed to determine the coordinates x , y , and z . First, the values of Γ_i ($i=1,2,3$) are evaluated at the known point. Then, a trial value is assumed for the axial coordinate x . From equations (D.10) and (D.11), the corresponding coordinates y and z may be obtained. Then, the radius $r^* = (y^2 + z^2)^{1/2}$ and the polar angle $\theta = \tan^{-1}(z/y)$ of the assumed intersection point may be computed. From the assumed value for x and the calculated value for θ ,

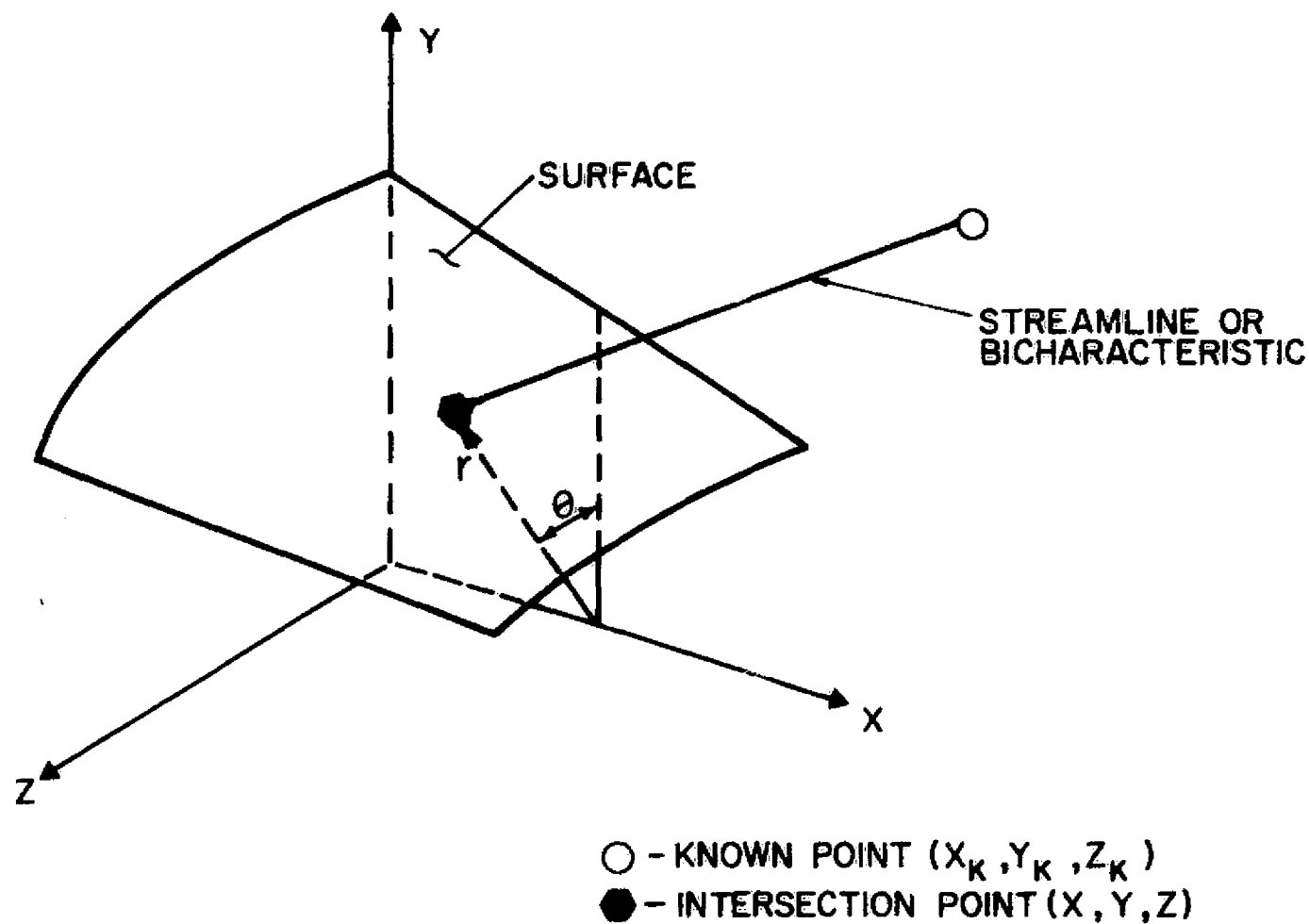


FIGURE D.2 INTERSECTION OF STREAMLINE OR BICHARACTERISTIC WITH A SURFACE.

the body radius r^{**} [determined from the tabular wall data or equation (D.1)] or the shock wave radius r^{**} [given by equation (D.5)] may be obtained. The difference between r^* and r^{**} is reduced to within a specified tolerance by employing a numerical relaxation technique (secant method) which iterates on x . Once convergence has been obtained, the values of Γ_i at the intersection point are computed using the trivariate interpolation method discussed in Appendix C. Appropriate averages of the values of Γ_i at the known point and the intersection point are then formed, and the entire process is repeated until overall convergence is obtained.

It should be noted that it is possible to use θ , instead of x , as the variable upon which the iterative scheme is based. The resulting formulation, however, is singular when the streamline or bicharacteristic lies in a meridional plane.

APPENDIX E

UNIT PROCESSES

1. INTRODUCTION

Computation of the flow field requires that a variety of unit processes be employed. These subalgorithms may be classified into four major types: interior point, solid boundary point, field-shock wave point, and body-shock wave point. Computation of the external flow field about the forebody portion of the centerbody requires using the basic versions of the first three aforementioned algorithms. Computation of the internal flow field, with its attendant reflected shock wave system, requires using the basic interior point and solid boundary point algorithms plus modified versions of these routines, as well as the other unit processes. All of the unit processes are presented in this appendix.

2. SUMMARY OF THE CHARACTERISTIC EQUATIONS

The equations for the characteristic surfaces and the compatibility equations valid along these surfaces are developed in Appendix B. A summary of the pertinent results is given below.

For steady three-dimensional supersonic flow, compatibility equations may be written which are valid when applied along either streamlines or bicharacteristics. A streamline is represented by the equation

$$dx_i = u_i dt \quad (i=1,2,3) \quad (E.1)$$

where x_i ($i=1,2,3$) denotes the three cartesian coordinates x , y , and z , respectively, u_i ($i=1,2,3$) denotes the corresponding velocity components u , v , and w , respectively, and t is the time of travel of a fluid particle along the streamline. The compatibility equations valid along a streamline are given by*

$$\frac{dP}{dt} - a^2 \frac{dp}{dt} = F_e \quad (E.2)$$

$$\frac{dP}{dt} + \rho u_i \frac{du_i}{dt} = u_i F_i \quad (E.3)$$

where P denotes the pressure, ρ is the density, a is the sonic speed, F_i ($i=1,2,3$) denotes the transport forcing terms in the x , y , and z component momentum equations, respectively, and F_e is the transport forcing term in the energy equation. The operator $d(\)/dt$ in equations (E.2) and (E.3) represents differentiation in the streamline direction. The forcing terms F_i and F_e are defined by equations (A.6) and (A.27), respectively.

A bicharacteristic, which is a ray or generator of the Mach cone, is represented by

$$dx_i = (u_i + c\alpha_i \cos\theta + c\beta_i \sin\theta)dt \quad (i=1,2,3) \quad (E.4)$$

where θ is a parametric angle denoting a particular element of the Mach cone and has the range $0 \leq \theta \leq 2\pi$, t is the time of travel of a fluid

* Repeated indices imply summation over the range of 1 to 3 unless otherwise noted.

particle along the streamline that is the axis of the Mach cone, and c is defined by

$$c^2 = q^2 a^2 / (q^2 - a^2) \quad (E.5)$$

where q is the velocity magnitude. The vectors α_i and β_j in equation (E.4) are parametric unit vectors with α_i , β_j , and u_i/q ($i=1,2,3$) forming an orthonormal set. The compatibility equation valid along a bicharacteristic is given by

$$\begin{aligned} \frac{dP}{dt} + \rho c (\alpha_i \cos\theta + \beta_i \sin\theta) \frac{du_i}{dt} = \Phi - \rho c^2 (\alpha_i \sin\theta \\ - \beta_i \cos\theta) (\alpha_j \sin\theta - \beta_j \cos\theta) \frac{\partial u_i}{\partial x_j} \end{aligned} \quad (E.6)$$

In equation (E.6), the operator $d(\)/dt$ represents differentiation in the bicharacteristic direction, and the parameter Φ is given by

$$\Phi = (c^2/a^2) (F_e - a n_i F_i) \quad (E.7)$$

where n_i is the i th component of the wave surface unit normal and is given by

$$n_i = (a/c) (cu_i/q^2 - \alpha_i \cos\theta - \beta_i \sin\theta) \quad (i=1,2,3) \quad (E.8)$$

In addition to the above relations, the following noncharacteristic relation is applied along a streamline

$$\frac{dP}{dt} = \sigma - \rho c^2 (\alpha_i \alpha_j + \beta_i \beta_j) \frac{\partial u_i}{\partial x_j} \quad (E.9)$$

where the operator $d(\)/dt$ represents differentiation in the streamline direction, and the parameter σ is given by

$$\sigma = (c^2/a^2)F_e - (c^2/q^2)(u_i F_i) \quad (E.10)$$

Equations (E.1) to (E.10) form the basis of the numerical integration method.

3. GENERAL COMMENTS CONCERNING THE UNIT PROCESSES

An inverse marching scheme is employed in the numerical algorithm. The solution is obtained on space-like planes of constant x , with the x -axis being the longitudinal axis of the centerbody and cowl. For the internal flow field, the solution is also obtained on the space curves which represent the intersection of the internal shock wave with the solid boundaries. These space curves are defined by the locus of shock wave solution points.

Except in the vicinity of a shock wave-solid boundary intersection, the distance between successive solution planes is determined by the application of the Courant-Friedrichs-Lewy (CFL) stability criterion, which is presented in Appendix F. The axial step in the vicinity of a shock wave-solid boundary intersection is controlled by special constraints which are also discussed in Appendix F.

Each of the unit processes is presented below. In general, a unit process is divided into a predictor step and a number of ensuing corrector steps. In most cases, a unit process employs an outer iterative loop for determination of the flow properties at the solution point, and an inner iterative loop (or loops) for location of bicharacteristic-initial-value plane intersection points, etc. The terms "inner" and "outer" are used in this context in the following discussions.

4. INTERIOR POINT UNIT PROCESS

Figure E.1 is a depiction of the computational network used in the determination of the solution for a typical interior point. Points (1) to (5) are located on the initial-value plane which is a plane of constant x on which the solution is known. Points (1) to (4) represent the intersection points of four rearward-running bicharacteristics with the initial-value plane, and point (5) is the intersection point of the streamline with this plane. Point (6) is the interior solution point, which is located at the intersection of the forward projection of the streamline with the solution plane. The axial (x) distance between the initial-value plane and the solution plane is determined by either the application of the CFL stability criterion, or, in the vicinity of a shock wave-solid boundary intersection, by the special constraints discussed in Appendix F.

Interpolated values of the three velocity components u , v , and w , the pressure P , and the density ρ are required at the bicharacteristic-initial-value plane intersection points, points (1) to (4) in Figure E.1. For this purpose, the following bivariate interpolation polynomial is employed

$$f(y,z) = a_1 + a_2y + a_3z + a_4yz + a_5y^2 + a_6z^2 \quad (E.11)$$

where $f(y,z)$ denotes a general function of the coordinates y and z . The coefficients a_i ($i=1$ to 6) in equation (E.11) are determined by a least squares fit of nine data points in the initial-value plane [point (5) and its eight immediate field point neighbors]. The detailed implementation of equation (E.11) is discussed in Appendix C.

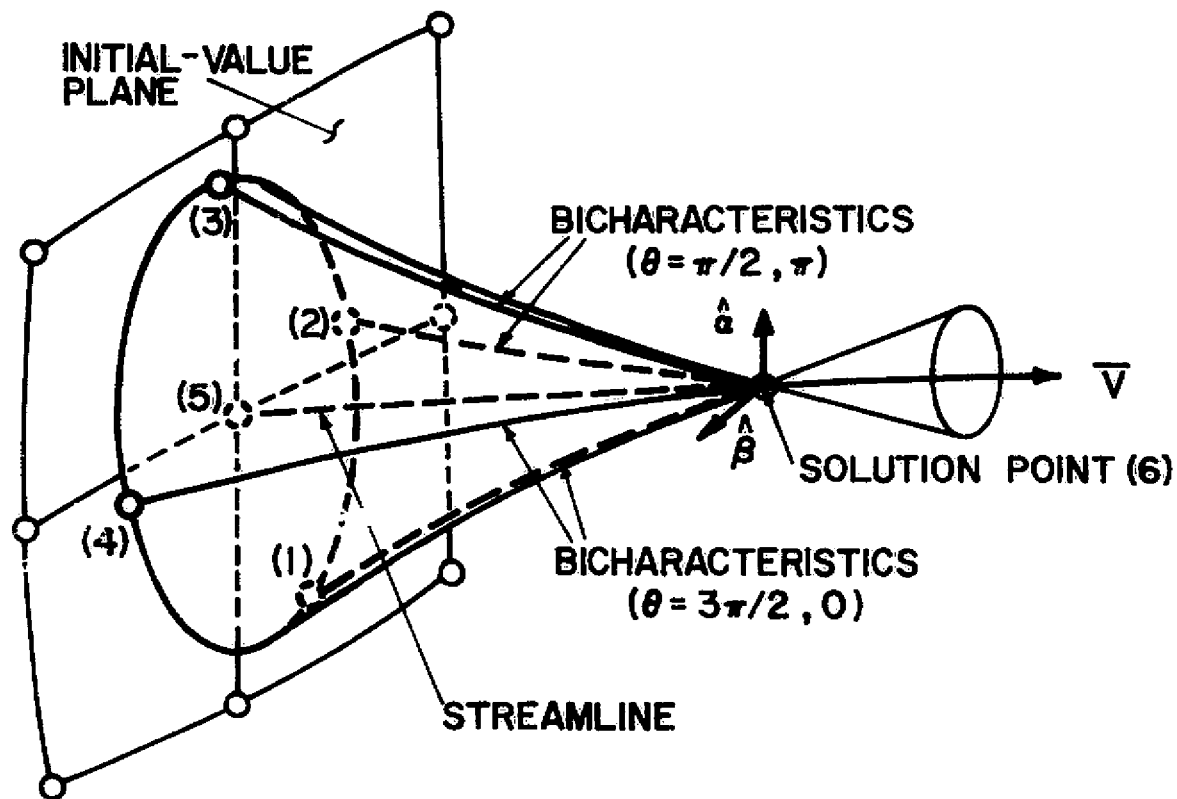


FIGURE E.1. INTERIOR POINT COMPUTATIONAL NETWORK

In addition to using interpolated values for the flow properties at points (1) to (4) in Figure E.1, interpolated values are also employed at point (5), the streamline base point, even though this is a field solution point. As shown by Ranson, et al. (9), this interpolation is required to produce a stable numerical scheme.

The interior point unit process is initiated by locating the solution point, point (6). This is accomplished by extending the streamline forward from point (5) to intersect the solution plane. The coordinates of point (6) are obtained using the following finite difference form of equation (E.1).

$$x_i(6) - x_i(5) = \frac{1}{2}[u_i(5) + u_i(6)][t(6) - t(5)] \quad (i=1,2,3) \quad (E.12)$$

In applying equation (E.12) for the predictor (first outer iteration), $u_i(6)$ is equated to $u_i(5)$, whereas, for the corrector (ensuing outer iteration), the previously obtained value of $u_i(6)$ is used.

Equation (E.12) is first applied for $i=1$ (i.e., the x-coordinate direction). The axial step $[x(6) - x(5)]$ is determined prior to the application of the unit process. Hence, the time parameter $[t(6) - t(5)]$ may be obtained. Then, equation (E.12) is applied for $i=2$ and $i=3$ to determine $y(6)$ and $z(6)$.

At this point, four bicharacteristics are extended backward from the solution point to intersect the initial-value plane. This is accomplished by applying the following finite difference form of equation (E.4).

$$x_i(6) - x_i(k) = \frac{1}{2} \{ [u_i(k) + u_i(6)] + [c(k) + c(6)] [\alpha_i \cos \theta(k) + \beta_i \sin \theta(k)] \} [t(6) - t(k)] \quad (i=1,2,3) \quad (E.13)$$

In equation (E.13), k denotes the bicharacteristic intersection points in Figure E.1 and has the values 1, 2, 3, and 4 corresponding to the $\theta(k)$ values of 0, $\pi/2$, π , and $3\pi/2$, respectively. The bicharacteristic intersection points are determined in an inner iterative loop. That is, for every outer iteration that is performed to determine the flow properties at point (6), a number of inner iterations are performed to locate points (1) to (4). On the first inner iteration of the predictor (the first outer iteration), $u_i(k)$ and $c(k)$ are equated to $u_i(5)$ and $c(5)$, respectively, for each of the four bicharacteristics. On ensuing inner and outer iterations, the flow properties previously obtained at each of the bicharacteristic intersection points are used. The flow properties at these points are determined by employing the bivariate interpolation polynomial given by equation (E.11). Moreover, as was done for equation (E.12), for the predictor (the first outer iteration), the flow properties at point (6) in equation (E.13) are set equal to those at point (5), whereas, for the corrector (ensuing outer iterations), previously computed values of the flow properties are used at the solution point.

Equation (E.13) is first applied for $i=1$ (i.e., the x -coordinate direction). The axial step $[x(6) - x(k)]$ is determined prior to the application of the unit process. Thus, the time parameter $[t(6) - t(k)]$ may be obtained for each of the four bicharacteristics. Then, equation (E.13) is applied for $i=2$ and $i=3$ to determine $y(k)$ and $z(k)$ for each bicharacteristic.

The parametric unit vectors α_i and β_i appearing in equation (E.13) are arbitrarily fixed at the solution point, point (6). Butler (24), in his original work, held α_i and β_i constant along a bicharacteristic but varied θ in order to insure that the bicharacteristic remained tangent to the Mach cone. Ransom, et al. (9) held θ constant along a bicharacteristic but varied α_i and β_i to satisfy this tangency condition. As noted by Cline, et al. (25), Butler (41) later realized that it is not necessary to satisfy the tangency condition in order to achieve second-order accuracy in the resulting overall numerical algorithm. As a consequence, in the present analysis, both θ and the unit vectors α_i and β_i are held constant along the bicharacteristics. For the external flow field integration, α_i and β_i are selected to straddle the projection of the pressure gradient in the initial-value plane. For the internal flow field integration, α_i and β_i are chosen to straddle the meridional plane.

Once the positions of and the flow properties at points (1) to (4) have been determined for a given outer iteration, the transport forcing functions F_x , F_y , F_z , and F_e are computed at each of these points and at the streamline base point, point (5), as described in Appendix G. Approximations for the transport forcing functions at point (6) are also made at this stage as described in Appendix G. The system of non-linear compatibility equations is then solved for the flow properties at point (6) as outlined below.

The compatibility equations valid along a streamline are given by equations (E.2) and (E.3). Writing those relations in finite difference form yields

$$\begin{aligned}
& [P(6) - P(5)]/[t(6) - t(5)] - \frac{1}{2}[a^2(5) + a^2(6)][\rho(6) \\
& - \rho(5)]/[t(6) - t(5)] = \frac{1}{2}[F_e(5) + F_e(6)] \quad (E.14)
\end{aligned}$$

$$\begin{aligned}
& [P(6) - P(5)]/[t(6) - t(5)] + \frac{1}{2}[\rho(5)u_i(5) + \rho(6)u_i(6)][u_i(6) \\
& - u_i(5)]/[t(6) - t(5)] = \frac{1}{2}[u_i(5)F_i(5) + u_i(6)F_i(6)] \quad (E.15)
\end{aligned}$$

The noncharacteristic equation, given by equation (E.9), is also applied along a streamline. Writing that equation in finite difference form gives

$$\begin{aligned}
& [P(6) - P(5)]/[t(6) - t(5)] = \frac{1}{2}[\sigma(5) + \sigma(6)] \\
& - \frac{1}{2}\rho(5)c^2(5)(\alpha_i\alpha_j + \beta_i\beta_j)\partial u_i/\partial x_j(5) \\
& - \frac{1}{2}\rho(6)c^2(6)(\alpha_i\alpha_j + \beta_i\beta_j)\partial u_i/\partial x_j(6) \quad (E.16)
\end{aligned}$$

In equation (E.16), σ is given by equation (E.10), and $\partial u_i/\partial x_j(k)$ denotes the appropriate partial derivative evaluated at point (k) in Figure E.1. Partial derivatives taken with respect to y and z are found by analytically differentiating equation (E.11). Partial derivatives taken with respect to x are then found by using the governing partial differential equations.

The compatibility equation valid along a bicharacteristic is given by equation (E.6). For θ values of 0 , $\pi/2$, π , and $3\pi/2$, equation (E.6) becomes

$$\frac{dP}{dt_1} + \rho c \alpha_i \frac{du_i}{dt_1} = \phi_1 - \rho c^2 \beta_i \beta_j \frac{\partial u_i}{\partial x_j} \quad (E.17)$$

$$\frac{dP}{dt_2} + \rho c \beta_i \frac{du_i}{dt_2} = \phi_2 - \rho c^2 \alpha_i \alpha_j \frac{\partial u_i}{\partial x_j} \quad (E.18)$$

$$\frac{dP}{dt_3} - \rho c \alpha_i \frac{du_i}{dt_3} = \phi_3 - \rho c^2 \beta_i \beta_j \frac{\partial u_i}{\partial x_j} \quad (E.19)$$

$$\frac{dP}{dt_4} - \rho c \beta_i \frac{du_i}{dt_4} = \phi_4 - \rho c^2 \alpha_i \alpha_j \frac{\partial u_i}{\partial x_j} \quad (E.20)$$

In equations (E.17) to (E.20), the operator $d(\)/dt_k$ denotes differentiation along the bicharacteristic corresponding to $\theta(k)$, and ϕ_k is determined from equation (E.7). Writing equations (E.17) to (E.20) in finite difference form yields

$$\begin{aligned} & [P(6) - P(1)]/[t(6) - t(1)] + \frac{1}{2}[\rho(1)c(1) \\ & + \rho(6)c(6)]\alpha_i[u_i(6) - u_i(1)]/[t(6) - t(1)] \\ & = \frac{1}{2}[\phi_1(1) + \phi_1(6)] - \frac{1}{2}\rho(1)c^2(1)\beta_i\beta_j\partial u_i/\partial x_j(1) \\ & - \frac{1}{2}\rho(6)c^2(6)\beta_i\beta_j\partial u_i/\partial x_j(6) \end{aligned} \quad (E.21)$$

$$\begin{aligned} & [P(6) - P(2)]/[t(6) - t(2)] + \frac{1}{2}[\rho(2)c(2) \\ & + \rho(6)c(6)]\beta_i[u_i(6) - u_i(2)]/[t(6) - t(2)] \\ & = \frac{1}{2}[\phi_2(2) + \phi_2(6)] - \frac{1}{2}\rho(2)c^2(2)\alpha_i\alpha_j\partial u_i/\partial x_j(2) \\ & - \frac{1}{2}\rho(6)c^2(6)\alpha_i\alpha_j\partial u_i/\partial x_j(6) \end{aligned} \quad (E.22)$$

$$\begin{aligned}
& [P(6) - P(3)]/[t(6) - t(3)] - \frac{1}{2}[\rho(3)c(3) \\
& + \rho(6)c(6)]\alpha_i[u_i(6) - u_i(3)]/[t(6) - t(3)] \\
& = \frac{1}{2}[\phi_3(3) + \phi_3(6)] - \frac{1}{2}\rho(3)c^2(3)\beta_i\beta_j\partial u_i/\partial x_j(3) \\
& - \frac{1}{2}\rho(6)c^2(6)\beta_i\beta_j\partial u_i/\partial x_j(6) \tag{E.23}
\end{aligned}$$

$$\begin{aligned}
& [P(6) - P(4)]/[t(6) - t(4)] - \frac{1}{2}[\rho(4)c(4) \\
& + \rho(6)c(6)]\beta_i[u_i(6) - u_i(4)]/[t(6) - t(4)] \\
& = \frac{1}{2}[\phi_4(4) + \phi_4(6)] - \frac{1}{2}\rho(4)c^2(4)\alpha_i\alpha_j\partial u_i/\partial x_j(4) \\
& - \frac{1}{2}\rho(6)c^2(6)\alpha_i\alpha_j\partial u_i/\partial x_j(6) \tag{E.24}
\end{aligned}$$

It was noted in Appendix B that only three wave surface compatibility relations are independent. To obtain three independent relations, linear combinations of equations (E.21) to (E.24) and the noncharacteristic relation, equation (E.16), are formed in such a manner as to algebraically eliminate the cross derivative terms at the solution point [i.e., terms containing $\partial u_i/\partial x_j(6)$]. Subtracting equation (E.23) from equation (E.21) yields

$$\begin{aligned}
& [P(6) - P(1)]/[t(6) - t(1)] - [P(6) - P(3)]/[t(6) - t(3)] \\
& + \frac{1}{2}[\rho(1)c(1) + \rho(6)c(6)]\alpha_i[u_i(6) - u_i(1)]/[t(6) - t(1)] \\
& + \frac{1}{2}[\rho(3)c(3) + \rho(6)c(6)]\alpha_i[u_i(6) - u_i(3)]/[t(6) - t(3)] \\
& = \frac{1}{2}[\phi_1(1) + \phi_1(6)] - \frac{1}{2}[\phi_3(3) + \phi_3(6)] \\
& - \frac{1}{2}\rho(1)c^2(1)\beta_i\beta_j\partial u_i/\partial x_j(1) + \frac{1}{2}\rho(3)c^2(3)\beta_i\beta_j\partial u_i/\partial x_j(3) \quad (E.25)
\end{aligned}$$

Subtracting equation (E.24) from equation (E.22) yields

$$\begin{aligned}
& [P(6) - P(2)]/[t(6) - t(2)] - [P(6) - P(4)]/[t(6) - t(4)] \\
& + \frac{1}{2}[\rho(2)c(2) + \rho(6)c(6)]\beta_i[u_i(6) - u_i(2)]/[t(6) - t(2)] \\
& + \frac{1}{2}[\rho(4)c(4) + \rho(6)c(6)]\beta_i[u_i(6) - u_i(4)]/[t(6) - t(4)] \\
& = \frac{1}{2}[\phi_2(2) + \phi_2(6)] - \frac{1}{2}[\phi_4(4) + \phi_4(6)] \\
& - \frac{1}{2}\rho(2)c^2(2)\alpha_i\alpha_j\partial u_i/\partial x_j(2) + \frac{1}{2}\rho(4)c^2(4)\alpha_i\alpha_j\partial u_i/\partial x_j(4) \quad (E.26)
\end{aligned}$$

Adding equations (E.21) and (E.22) and subtracting equation (E.16) from the sum yields

$$\begin{aligned}
& [P(6) - P(1)]/[t(6) - t(1)] + [P(6) - P(2)]/[t(6) - t(2)] \\
& - [P(6) - P(5)]/[t(6) - t(5)] \\
& + \frac{1}{2}[\rho(1)c(1) + \rho(6)c(6)]\alpha_i[u_i(6) - u_i(1)]/[t(6) - t(1)] \\
& + \frac{1}{2}[\rho(2)c(2) + \rho(6)c(6)]\beta_i[u_i(6) - u_i(2)]/[t(6) - t(2)]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}[\phi_1(1) + \phi_1(6)] + \frac{1}{2}[\phi_2(2) + \phi_2(6)] - \frac{1}{2}[\sigma(5) + \sigma(6)] \\
&- \frac{1}{2}\rho(1)c^2(1)\beta_i\beta_j\partial u_i/\partial x_j(1) - \frac{1}{2}\rho(2)c^2(2)\alpha_i\alpha_j\partial u_i/\partial x_j(2) \\
&+ \frac{1}{2}\rho(5)c^2(5)(\alpha_i\alpha_j + \beta_i\beta_j)\partial u_i/\partial x_j(5) \quad (E.27)
\end{aligned}$$

Equations (E.14), (E.15), (E.25), (E.26), and (E.27) are the five finite difference equations which are used to solve for the flow properties $u(6)$, $v(6)$, $w(6)$, $P(6)$, and $\rho(6)$. Since these equations are nonlinear, an iterative scheme is required to obtain the solution. On the first outer iteration (the predictor), all of the flow properties at point (6) appearing in the coefficients of the derivatives in the above set of equations are set equal to the respective properties at point (5). This produces a system of simultaneous linear equations which is solved using a Gaussian elimination method with complete pivoting (40). On ensuing corrector applications (outer iterations), previously computed values for the flow properties at point (6) are employed in the scheme. This method is similar to the Euler predictor-corrector algorithm used to obtain the solution for initial-value problems for ordinary differential equations, and can be shown to have second-order accuracy either by direct numerical calculation (9) or by substituting an exact solution into the difference equations and expanding the resulting terms in a Taylor series and thereby determining the truncation error. The iterative scheme is terminated when all five flow properties at point (6) have converged to within specified tolerances.

5. SOLID BOUNDARY POINT UNIT PROCESS

Figure E.2 is a depiction of the computational network used in determining the solution for a typical point on a solid boundary. The point notation used in Figure E.2 is the same as that used in Figure E.1 (interior point scheme). In this unit process, however, point (4), corresponding to the bicharacteristic with $\theta = 3\pi/2$, falls outside of the flow field and cannot be employed. Furthermore, the streamline points (5) and (6) lie on the stream surface formed by the solid boundary. The formulations used for representing the solid boundaries are presented in Appendix D.

The boundary condition used in this unit process is simply that the flow be tangent to the surface of the boundary at the solution point, point (6) in Figure E.2. Let n_{bi} ($i=1,2,3$) denote the x, y, and z components, respectively, of the outward unit normal to the solid boundary surface. Then, the flow tangency boundary condition may be written as

$$u_i(6) n_{bi}(6) = 0 \quad (E.28)$$

The solid boundary point unit process is virtually identical to the interior point unit process, except that the wave surface compatibility equation valid along the bicharacteristic corresponding to $\theta = 3\pi/2$ is not employed. That equation is replaced by equation (E.28). Thus, the system of compatibility equations used for determining the solution at a solid boundary point consists of equations (E.14), (E.15), (E.25), (E.27), and (E.28). This system of equations is solved using the same iterative scheme that was employed in the interior point solution.

The location of the solution point, point (6) in Figure E.2, obtained by applying the finite difference form of the streamline equation, equation (E.12), is adjusted along the projection of the body normal in the solution plane so that the solution point lies on the solid boundary. The orientation of the parametric unit vectors α_i and β_i is selected such that $\beta_i = -n_{bi}$ ($i=1,2,3$), and α_i ($i=1,2,3$) is found by employing the orthonormal relations between α_i , β_i , and u_i/q . This selection for the reference vector set produces a computational network in which the bicharacteristics corresponding to $\theta = 0$, $\pi/2$, and π intersect the initial-value plane for convex boundaries. For concave boundaries, those bicharacteristics intersect an extrapolation of the initial-value plane (the required extrapolation is assumed to have an error third-order in step size). The bicharacteristics corresponding to $\theta = 0$ and π lie in the elemental plane which is tangent to the solid boundary at point (6).

6. BOW SHOCK WAVE POINT UNIT PROCESS

A depiction of the computational network used in determining the solution for a typical bow shock wave point is given in Figure E.3. A segment of the shock wave surface extending from the initial-value plane to the solution plane is shown in this figure. The space curve (A) is defined by the intersection of the shock wave with the initial-value plane, whereas, space curve (B) is defined by the intersection of the shock wave with the solution plane. The axial distance between the initial-value plane and the solution plane is determined by the application of the CFL stability criterion.

The bow shock wave solution point is denoted by point (2) in Figure E.3. The flow properties upstream of the shock wave are known

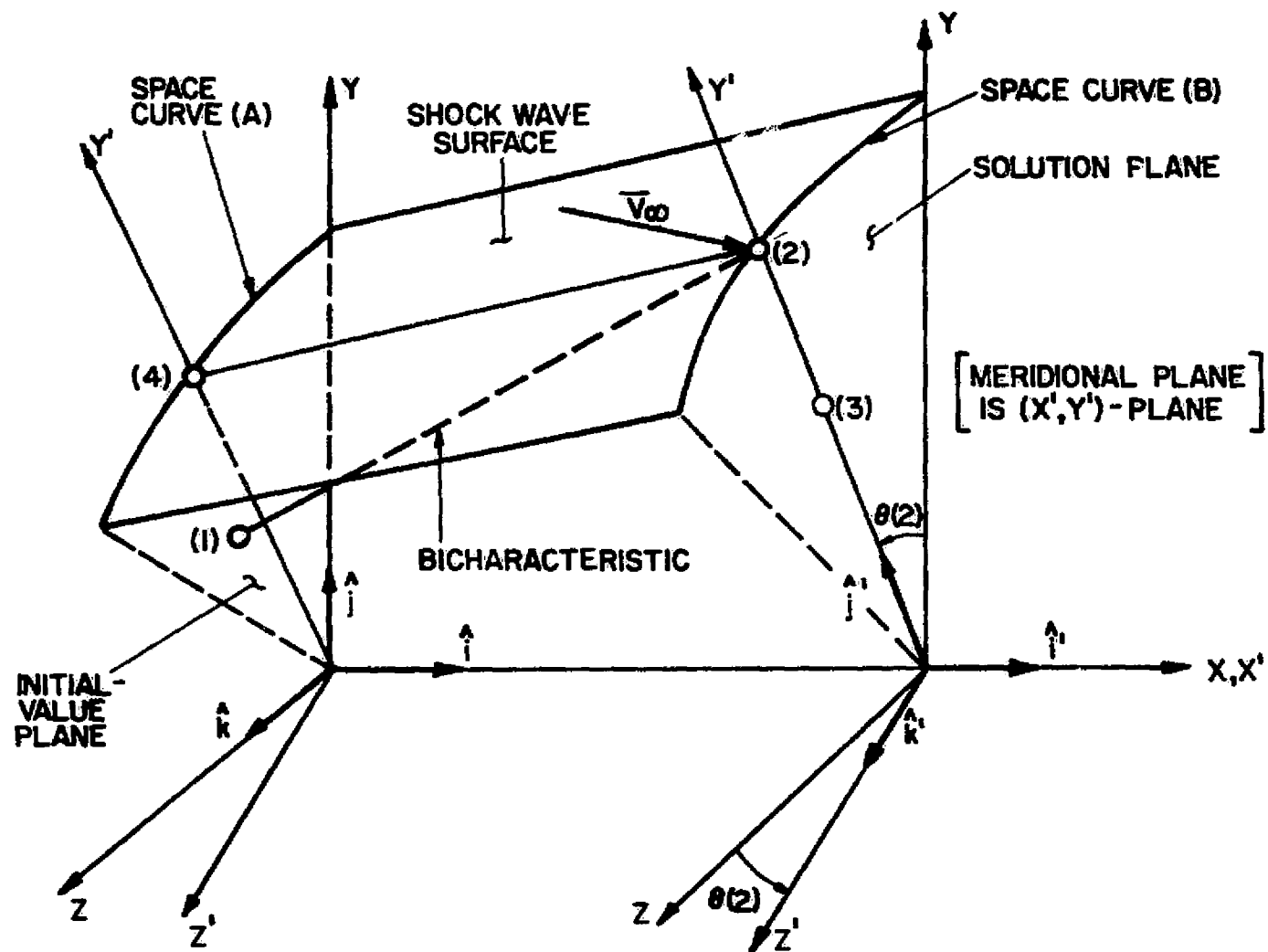


FIGURE E.3. BOW SHOCK WAVE POINT COMPUTATIONAL NETWORK

a priori. Hence, in the following discussion, the flow properties $u(2)$, $v(2)$, $w(2)$, $P(2)$, and $\rho(2)$ refer to the properties at point (2) downstream of the shock wave. Point (1) is the intersection point of a rearward-running bicharacteristic with the initial-value plane. This bicharacteristic is extended backward from the solution point. Point (3) is an interior point in the solution plane which is used to define the meridional plane in which the shock wave solution point lies. Point (4) is the intersection point of space curve (A) with the meridional plane which passes through points (2) and (3).

In this unit process, a local cartesian coordinate system is employed for the description of the orientation of the local shock wave surface. This local coordinate system has coordinates x' , y' , and z' , where x' is coincident with the x -axis, y' is in the radial direction corresponding to the meridional plane which subtends an angle θ with the (x,y) -plane, and z' is normal to the (x',y') -plane (see Figure E.3). The unit vectors in the x , y , and z directions are denoted by $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively, whereas, the unit vectors in the x' , y' , and z' directions are denoted by $\hat{i}'$, $\hat{j}'$, and $\hat{k}'$, respectively. A vector quantity $\bar{A}$ may be represented in these coordinate systems by

$$\bar{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad (E.29)$$

$$\bar{A} = A_{x'} \hat{i}' + A_{y'} \hat{j}' + A_{z'} \hat{k}' \quad (E.30)$$

The relationships between the respective components in equations (E.29) and (E.30) are given by

$$A_{x'} = A_x \quad (E.31)$$

$$A_{y'} = A_y \cos\theta + A_z \sin\theta \quad (\text{E.32})$$

$$A_{z'} = A_z \cos\theta - A_y \sin\theta \quad (\text{E.33})$$

$$A_x = A_{x'} \quad (\text{E.34})$$

$$A_y = A_{y'} \cos\theta - A_{z'} \sin\theta \quad (\text{E.35})$$

$$A_z = A_{z'} \cos\theta + A_{y'} \sin\theta \quad (\text{E.36})$$

The orientation of the local shock wave surface is specified by a set of unit vectors referenced to the (x', y', z') -system. This set of unit vectors, illustrated in Figure E.4, consists of a unit vector $\hat{n}_s$ which is normal to the shock wave surface and two unit vectors $\hat{l}$ and $\hat{t}$ which are tangent to this surface. The tangential unit vector $\hat{t}$ lies in the meridional plane $[(x', y')\text{-plane}]$, subtends an angle ϕ with the x' -axis, and is defined by the intersection of the shock wave with the meridional plane at point (P). The tangential unit vector $\hat{l}$ lies in the transverse plane $[(y', z')\text{-plane}]$, subtends an angle α with the z' -axis, and is defined by the intersection of the shock wave with the transverse plane at point (P). The tangential vectors $\hat{t}$ and $\hat{l}$ are therefore given by

$$\hat{t} = \cos\phi \hat{i}' + \sin\phi \hat{j}' \quad (\text{E.37})$$

$$\hat{l} = \sin\alpha \hat{j}' + \cos\alpha \hat{k}' \quad (\text{E.38})$$

The shock wave normal unit vector, denoted by $\hat{n}_s$, is given by

$$\hat{n}_s = \hat{l} \times \hat{t} / |\hat{l} \times \hat{t}| \quad (\text{E.39})$$

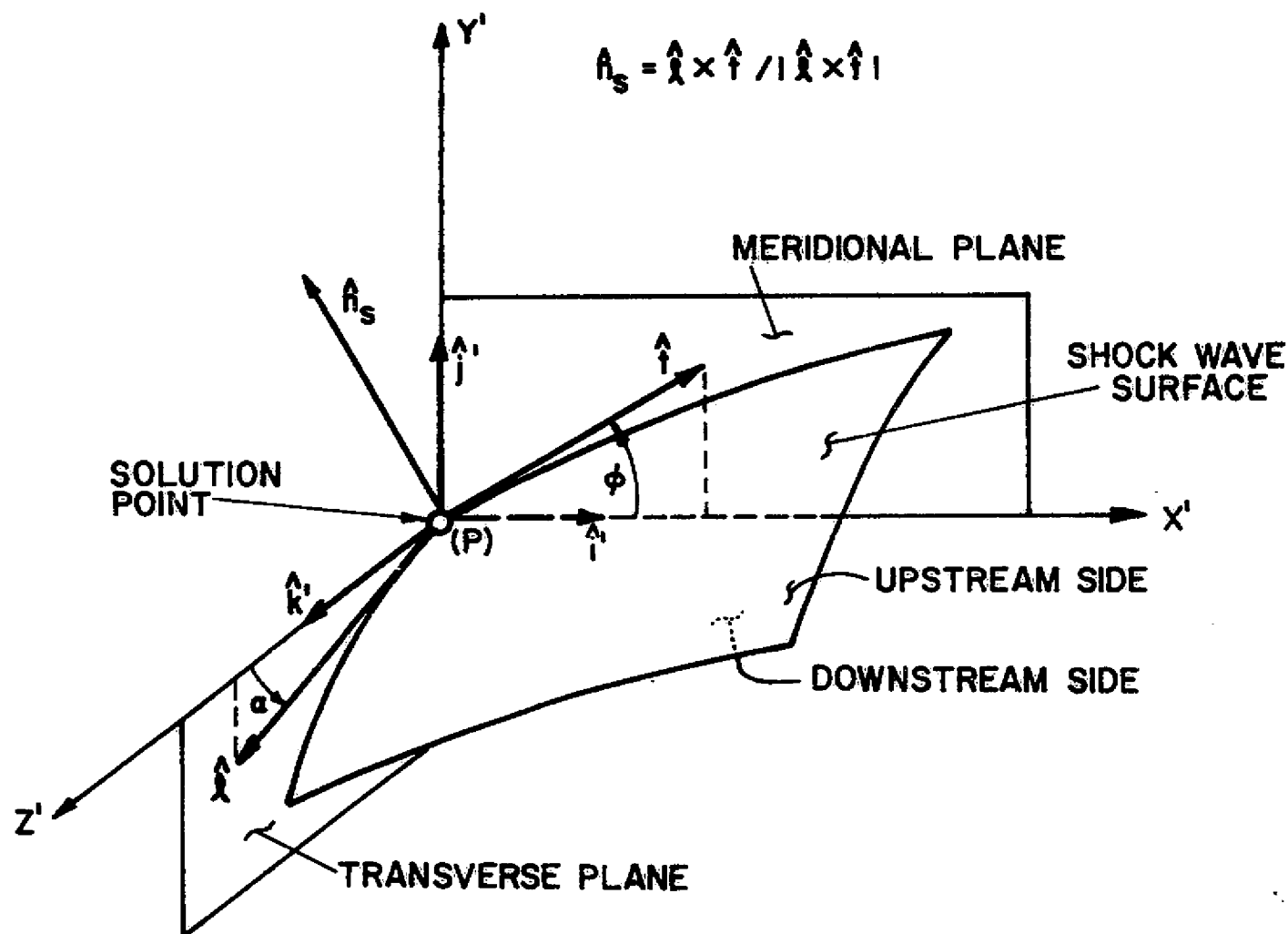


FIGURE E.4. UNIT VECTORS FOR SPECIFICATION OF SHOCK WAVE SURFACE ORIENTATION

The interior point and solid boundary point unit processes achieve second-order accuracy by using local iteration. In local iteration, a corrector application employs previously determined flow property values at the solution point, but does not require using flow property values at other points in the solution plane. The shock wave point unit process, however, requires that global iteration be performed in order to achieve second-order accuracy. In global iteration, a corrector application employs previously determined flow property values not only at the solution point, but also at neighboring points in the solution plane. As a consequence, before a corrector application in global iteration can be performed, the entire solution plane (or at least an appropriate section of it) must be determined by a prior calculation. In practice, since the interior point and solid boundary point schemes require local iteration only, the interior point and solid boundary points are computed first. Then, a prediction for each shock wave solution point is made, thereby giving a tentative solution for all of the shock wave points. Then, a global iteration is conducted for the shock wave solution points using the previously determined field points in the solution plane. In the following discussion, the term "predictor" will refer to the first application of the shock wave point unit process used to obtain an initial estimate of the solution without using field point data in the solution plane. The term "global corrector" will refer to the application of the shock wave point unit process which uses field point data in the solution plane. The shock wave point unit process is now outlined.

The shock wave point unit process is initiated by locating the solution point, point (2) in Figure E.3. The meridional plane in which the solution point lies is arbitrarily selected to contain point (3). Point (3) is the interior solution point adjacent to the shock wave surface whose location is determined prior to the application of the shock wave point unit process. The angle subtended by a meridional plane and the (x,y)-plane is denoted by θ . Then

$$\theta(2) = \theta(3) = \tan^{-1}[z(3)/y(3)] \quad (E.40)$$

Denote the radial position of a point by r . Then the radial position of point (2) is obtained from

$$r(2) = r(4) + [x(2) - x(4)] \tan \left\{ \frac{1}{2}[\phi(2) + \phi(4)] \right\} \quad (E.41)$$

where $[x(2) - x(4)]$ is the axial distance between the initial-value plane and the solution plane and is determined by the CFL stability criterion. On the first application of equation (E.41), the shock wave angle $\phi(2)$ is equated to $\phi(4)$, whereas, on ensuing applications, the previously determined value of $\phi(2)$ is used. At point (4), the radial position $r(4)$ and shock wave angle $\phi(4)$ are determined by interpolation using the quadratic univariate formulae

$$r(\theta) = a_1 + a_2\theta + a_3\theta^2 \quad (E.42)$$

$$\phi(\theta) = b_1 + b_2\theta + b_3\theta^2 \quad (E.43)$$

In equations (E.42) and (E.43), the coefficients a_i ($i=1,2,3$) and b_i ($i=1,2,3$) are determined by fitting these expressions to three local shock wave solution points on space curve (A) as described in Appendix C.

For the case of axisymmetric flow, or on a plane of flow symmetry in three-dimensional flow, point (4) coincides with a previously determined shock wave solution point so the interpolation would not be required. In general, however, point (4) does not coincide with a known point so the interpolation is necessary.

After the solution point has been located, the shock wave normal unit vector $\hat{n}_s$ at the solution point is found by forming the normalized cross product of the tangential unit vectors $\hat{x}$ and $\hat{t}$ [see equation (E.39)]. The tangential vector $\hat{t}$ is obtained by using the current value of $\phi(2)$ in equation (E.37). The tangential vector $\hat{x}$ is obtained by using the current value of $\alpha(2)$ in equation (E.38). For either space curve (A) or space curve (B), the value of $\alpha(2)$ may be obtained from

$$\alpha(2) = \tan^{-1} \left(\frac{1}{r} \frac{dr}{d\theta} \right) \bigg|_{\theta(2)} \quad (E.44)$$

For a predictor application, the analytical form of $r(\theta)$ used in equation (E.44) is given by equation (E.42) applied along space curve (A), whereas, for a global corrector application, $r(\theta)$ is obtained from equation (E.42) applied along space curve (B).

After the shock wave normal unit vector has been determined, the local Hugoniot equations may be applied across the shock wave, thereby yielding a solution for the flow properties $u(2)$, $v(2)$, $w(2)$, $P(2)$, and $\rho(2)$. In general, the local Hugoniot equations take the form (5)

$$\rho_u \tilde{V}_{nu} = \rho_d \tilde{V}_{nd} \quad (E.45)$$

$$P_u + \rho_u \tilde{V}_{nu}^2 = P_d + \rho_d \tilde{V}_{nd}^2 \quad (E.46)$$

$$\tilde{V}_{tu} = \tilde{V}_{td} \quad (E.47)$$

$$\tilde{V}_{\ell u} = \tilde{V}_{\ell d} \quad (E.48)$$

$$h_u + q_u^2/2 = h_d + q_d^2/2 \quad (E.49)$$

$$h = h(P, \rho) \quad (E.50)$$

In equations (E.45) to (E.50), h is the enthalpy per unit mass, q is the velocity magnitude ($q^2 = u^2 + v^2 + w^2$), $\tilde{V}_n$ is the velocity component in the $-\hat{n}_s$ direction, $\tilde{V}_t$ is the velocity component in the $\hat{t}$ direction, $\tilde{V}_\ell$ is the velocity component in the $\hat{\ell}$ direction, and the subscripts u and d denote the properties on the upstream and downstream sides of the shock wave, respectively. Equations (E.45) to (E.50) are solved simultaneously for the downstream flow properties. To obtain the velocity components $\tilde{V}_{nu}$, $\tilde{V}_{tu}$, and $\tilde{V}_{\ell u}$, the upstream velocity vector is first transformed from the (x, y, z) -system to the (x', y', z') -system using equations (E.31) to (E.33), after which the appropriate dot products are formed with $-\hat{n}_s$, $\hat{t}$, and $\hat{\ell}$. Similarly, the downstream velocity components $\tilde{V}_{nd}$, $\tilde{V}_{td}$, and $\tilde{V}_{\ell d}$ are transformed back to the (x, y, z) -system after the local Hugoniot equations have been applied.

In the computer program, the local Hugoniot equations are contained in a separate subroutine. The assumed thermodynamic model is that of a thermally and calorically perfect gas. Other thermodynamic models may be used by suitably modifying the existing subroutine or replacing it. For the assumed model of a thermally and calorically perfect gas, the pressure ratio across the shock wave is given by

$$\frac{P_d}{P_u} = \frac{2\gamma}{\gamma+1} M_{nu}^2 - \frac{\gamma-1}{\gamma+1} \quad (E.51)$$

where M_{nu} is the incident normal Mach number given by

$$M_{nu} = \tilde{V}_{nu}/a_u \quad (E.52)$$

and γ is the specific heat ratio. Using the result of equation (E.51), the density ratio across the shock wave is given by

$$\frac{\rho_d}{\rho_u} = \frac{(\gamma+1)/(\gamma-1) + (P_u/P_d)}{1 + [(\gamma+1)/(\gamma-1)](P_u/P_d)} \quad (E.53)$$

With the downstream pressure and density determined, the downstream normal velocity component $\tilde{V}_{nd}$ may be obtained from equation (E.46), and the tangential downstream velocity components $\tilde{V}_{td}$ and $\tilde{V}_{ld}$ may be computed from equations (E.47) and (E.48). Transformation of the downstream velocity components back into the (x,y,z)-system yields the required flow properties at the solution point.

At this stage, a rearward-running bicharacteristic is extended from the solution point, point (2), back to the initial-value plane, intersecting this plane at point (1), as illustrated in Figure E.3. This is accomplished by employing the following finite difference form of equation (E.4) evaluated for the parametric angle $\theta = \pi/2$.

$$x_i(2) - x_i(1) = \frac{1}{2} \left\{ [u_i(1) + u_i(2)] + [c(1) + c(2)] \beta_i \right\} [t(2) - t(1)] \quad (i=1,2,3) \quad (E.54)$$

As in the interior point and solid boundary point schemes, an inner iteration is performed to locate point (1). On the first application

of equation (E.54), the flow properties at point (1) are equated to those at point (2), whereas, on ensuing applications, previously obtained values of the flow properties at point (1) are used. The flow property values at point (1) are found by employing the bivariate interpolation polynomial given by equation (E.11). The coefficients in equation (E.11) are obtained by a least squares fit of nine data points in the initial-value plane using a boundary-type stencil as described in Appendix C.

Equation (E.54) is first applied for $i=1$ (i.e., the x-coordinate direction). Since the axial step $[x(2) - x(1)]$ is known from the application of the CFL stability criterion, the time parameter $[t(2) - t(1)]$ may be determined. Then, equation (E.54) is applied for $i=2$ and $i=3$ to determine $y(1)$ and $z(1)$. For axisymmetric flow, or for a plane of flow symmetry in three-dimensional flow, point (1) lies in the meridional plane which contains points (2) and (3). In general, however, for other flow situations, point (1) lies outside of this plane.

The orientation of the parametric unit vector β_i in equation (E.54) is arbitrarily selected such that

$$\beta_3/\beta_2 = \tan[\theta(2)] \quad (E.55)$$

This relation, in conjunction with the orthonormality conditions

$$\beta_i u_i(2) = 0 \quad (E.56)$$

$$\beta_i \beta_i = 1 \quad (E.57)$$

allows the values of β_i ($i=1,2,3$) to be determined. Since equation

(E.57) is a quadratic equation, a multiplicity of roots exist for the β_i ($i=1,2,3$). The roots are chosen such that point (1) lies underneath the shock wave in the initial-value plane. Once the values of β_i ($i=1,2,3$) are determined, the values of α_i ($i=1,2,3$) are found through use of the orthogonality relation between α_i , β_i , and u_i/q (i.e., $\hat{\alpha} = \hat{\beta} \times \bar{V}/q$).

After the position of and the flow properties at point (1) have been determined, the transport forcing functions F_x , F_y , F_z , and F_e are computed at point (1) as described in Appendix G. Approximations for the transport forcing functions are also made at point (2) at this time as described in Appendix G.

At this stage, the wave surface compatibility equation corresponding to the parametric angle $\theta = \pi/2$ is applied between points (1) and (2). From equation (E.6), the appropriate equation is

$$\frac{dP}{dt} + \rho c \beta_i \frac{du_i}{dt} = \phi_{\pi/2} - \rho c^2 \alpha_i \alpha_j \frac{\partial u_i}{\partial x_j} \quad (E.58)$$

where $\phi_{\pi/2}$ is obtained from equation (E.7) for the parametric angle $\theta = \pi/2$. Writing equation (E.58) in finite difference form, solving for the pressure at point (2), and denoting this pressure by $P^*(2)$, the following equation is obtained.

$$\begin{aligned} P^*(2) = & P(1) + \frac{1}{2}[\phi_{\pi/2}(1) + \phi_{\pi/2}(2)][t(2) - t(1)] \\ & - \frac{1}{2}[\rho(1)c^2(1)\alpha_i\alpha_j\partial u_i/\partial x_j(1) \\ & + \rho(2)c^2(2)\alpha_i\alpha_j\partial u_i/\partial x_j(2)][t(2) - t(1)] \\ & - \frac{1}{2}[\rho(1)c(1) + \rho(2)c(2)]\beta_i[u_i(2) - u_i(1)] \end{aligned} \quad (E.59)$$

Note that the cross-derivative terms $[\partial u_i / \partial x_j(k)]$ in equation (E.59) appear at both point (1) in the initial-value plane and at point (2) in the solution plane. In general, these terms can be evaluated by employing equation (E.11) fit to nine data points in the appropriate plane, differentiating this expression analytically to obtain partial derivatives with respect to y and z , and then using the governing partial differential equations to obtain the required partial derivatives with respect to x . On the predictor application of the shock wave point unit process, the flow property field in the solution plane is not known, so the cross-derivatives at point (2) are set equal to those at point (1). On a global corrector application of the shock wave point unit process, the cross derivatives at point (2) are evaluated in the manner just described.

The pressure $P(2)$ is calculated from the local Hugoniot equations. The pressure $P^*(2)$ is calculated from equation (E.59). The difference between $P(2)$ and $P^*(2)$ is driven to within a specified tolerance of zero by employing a one-dimensional secant iteration scheme which iterates on the shock wave angle $\phi(2)$. Two initial estimates of $\phi(2)$ are required to initiate the subiteration.

The shock wave point unit process is first applied as a predictor for each shock wave solution point. In this application, the value of α used in equation (E.38) is obtained by curve fitting points along space curve (A), and the cross-derivative terms at the shock wave solution point are equated to those terms at the bicharacteristic base

point in the initial-value plane, point (1). After a tentative solution is obtained for all of the shock wave points, a number of global corrector applications are performed. Here, the value of α used in equation (E.38) is based on data along space curve (B), and the cross-derivative terms at the shock wave solution point are evaluated at that point. The resulting overall scheme has second-order accuracy when the global correction is performed. The global iteration is terminated when successive values of α have converged at each of the shock wave solution points.

In the course of the program development, an alternative algorithm to the one just presented was devised in an attempt to compute the bow shock wave solution points. In this alternative scheme, a multiplicity of bicharacteristics were used, and, like the interior point or solid boundary point unit processes, linear combinations of the wave surface compatibility equations were formed as to algebraically eliminate the cross-derivative terms at the solution point. A two-dimensional Newton-Raphson method was devised for determining the angles ϕ and α explicitly, and second-order accuracy was achieved without resorting to global correction. This scheme was successful in computing axisymmetric flows, but an apparent instability arose when attempting to compute three-dimensional flow fields.

7. SOLID BODY-SHOCK WAVE POINT UNIT PROCESS

The solid body-shock wave point unit process is used to determine the flow properties downstream of the shock wave at a point where the shock wave intersects a solid boundary. This unit process is used to determine the solution for the points on the cowl on the downstream side

of the cowl lip shock wave, and for the points on the centerbody or cowl on the downstream side of an internal reflected shock wave. The method of computation is essentially the same for either application and is discussed below. The solution points on the downstream side of the incident shock wave at an internal shock wave reflection are computed using the field-shock wave point unit process which is presented later.

A depiction of the computational network used in the solid body-shock wave point unit process is presented in Figure E.5. A typical solid body-shock wave solution point is denoted by point (P) in this figure. At point (P), the outward unit normal vector to the solid boundary is denoted by $\hat{n}_b$. The locus of solid body-shock wave solution points represents the intersection of the shock wave with the solid boundary, and defines space curve (A) in Figure E.5. The intersection of the shock wave with the meridional plane passing through point (P) is denoted by space curve (B). The tangential unit vectors to space curves (A) and (B) at point (P) are denoted by $\hat{e}$ and $\hat{t}$, respectively. The unit normal vector to the shock wave at point (P) is denoted by $\hat{n}_s$.

As was done for the bow shock wave point unit process, the unit vectors $\hat{e}$, $\hat{t}$, and $\hat{n}_s$ are referenced to a local cartesian coordinate system (x', y', z') , where again x' is coincident with the x -axis, y' is in the radial direction along the meridian which subtends the angle μ with the (x, y) -plane, and z' is normal to the (x', y') -plane. The relations between the components of a vector in the (x, y, z) -system and in the (x', y', z') -system are given by equations (E.31) to (E.36). As in the bow shock wave point unit process, the tangential unit vector $\hat{t}$ lies

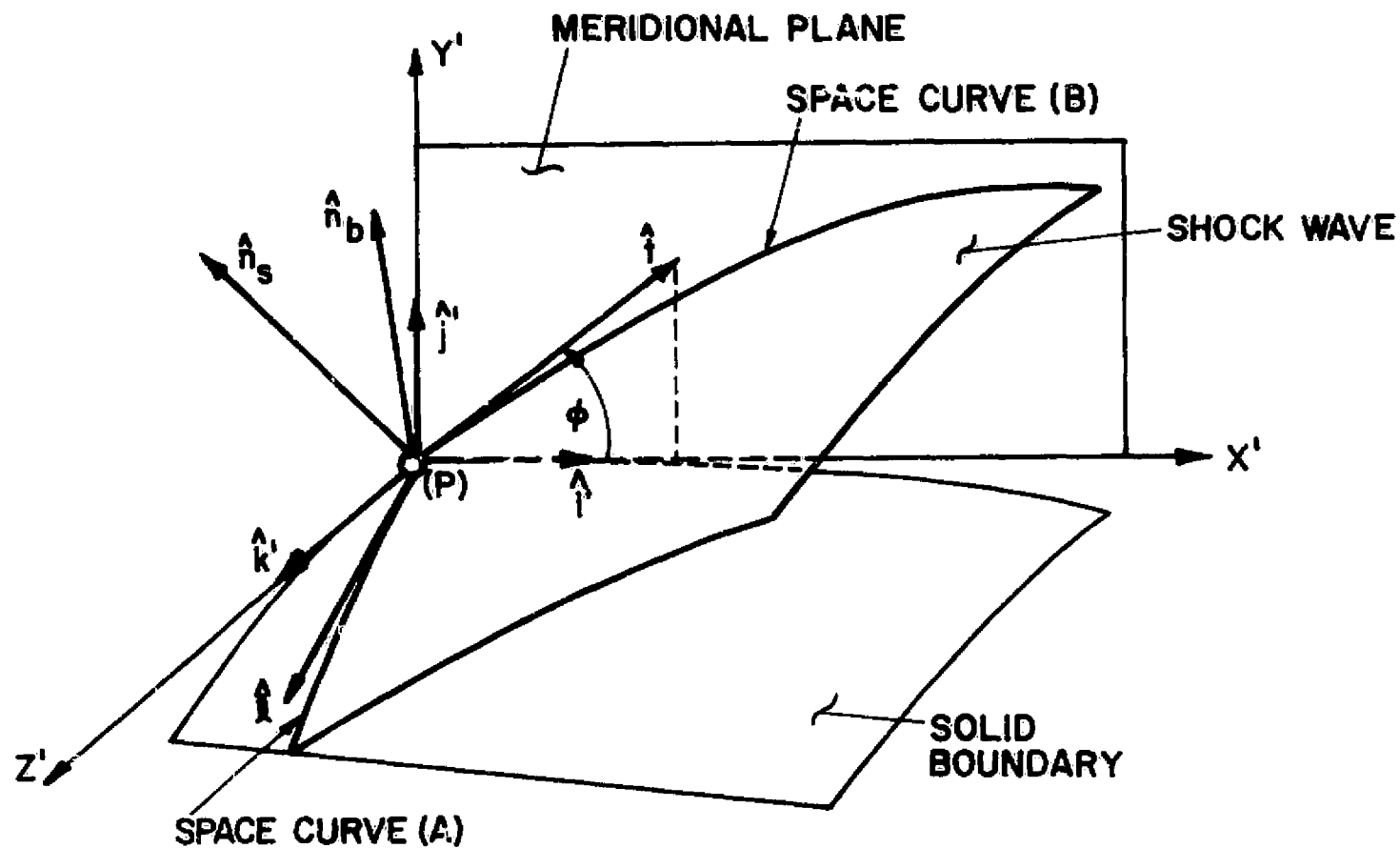


FIGURE E.5. SOLID BODY-SHOCK WAVE POINT COMPUTATIONAL NETWORK

in the meridional plane $[(x',y')\text{-plane}]$ and subtends the angle ϕ with the x' -axis. Hence,

$$\hat{t} = \cos\phi \hat{i}' + \sin\phi \hat{j}' \quad (\text{E.60})$$

Unlike the bow shock wave point unit process, however, the tangential unit vector $\hat{\ell}$ does not, in general, lie in the transverse plane $[(y',z')\text{-plane}]$, but rather it may have a nonzero x' -component. This tangential vector along space curve (A) may be represented by

$$\hat{\ell} = \frac{dx'}{ds} \hat{i}' + \frac{dy'}{ds} \hat{j}' + \frac{dz'}{ds} \hat{k}' \quad (\text{E.61})$$

where ds is the differential arc length given by

$$(ds)^2 = (dx')^2 + (dy')^2 + (dz')^2 \quad (\text{E.62})$$

The derivatives in equation (E.61) are obtained by analytically differentiating the expressions

$$x'(\theta) = a_1 + a_2\theta + a_3\theta^2 \quad (\text{E.63})$$

$$y'(\theta) = b_1 + b_2\theta + b_3\theta^2 \quad (\text{E.64})$$

$$z'(\theta) = c_1 + c_2\theta + c_3\theta^2 \quad (\text{E.65})$$

In equations (E.63) to (E.65), the coefficients a_i , b_i , and c_i ($i=1,2,3$) are obtained by fitting the respective expressions to three points on space curve (A) as described in Appendix C. For the cowl lip shock wave points, space curve (A) is defined by the cowl lip itself since the shock wave is assumed to be attached to the cowl lip. In this case, the x' -component in equation (E.61) is identically zero,

and, as a consequence, $\hat{\ell}$ lies in the transverse plane. Furthermore, if the cowl is axisymmetric, the y' -component is also identically zero. Alternatively, for computing the downstream properties at a reflected internal shock wave, space curve (A) is defined by the intersection of the incident shock wave with the solid boundary. Except for an axisymmetric flow field, or for a point on a plane of flow symmetry in three-dimensional flow, the x' -component in equation (E.61) is nonzero. With the tangential unit vectors determined, the shock wave normal unit vector $\hat{n}_s$ is obtained from equation (E.39).

The solid body-shock wave point unit process is initiated by determining the body normal unit vector $\hat{n}_b$ and the tangential unit vector $\hat{\ell}$ at point (P), expressing both of these vectors in the (x',y',z') -system. Then, an initial estimate is made for the value of ϕ in equation (E.60), and, by use of equation (E.39), the shock wave normal unit vector is obtained. In exactly the same manner as was done in the bow shock wave point unit process, the downstream flow properties at point (P) are computed by use of equations (E.45) to (E.53). At this stage, the velocity normal to the body V_{nb} at point (P) is computed from the equation

$$V_{nb} = u'_d n_{bx'} + v'_d n_{by'} + w'_d n_{bz'} \quad (E.66)$$

where u'_d , v'_d , and w'_d are the downstream velocity components at point (P), and $n_{bx'}$, $n_{by'}$, and $n_{bz'}$ are the components of the body normal unit vector, both vectors being expressed in terms of the (x',y',z') coordinates. The body normal velocity V_{nb} is reduced to within a specified tolerance of zero by varying the angle ϕ using a one-dimensional secant iteration procedure. Two initial estimates of ϕ are

required for starting the iterative procedure. Once convergence has been obtained, the downstream velocity components are transformed back into the (x,y,z) -coordinates using equations (E.34) to (E.36).

In the course of the program development, an alternative algorithm to the one just presented was devised to compute the solid body-shock wave points. That algorithm determined the shock normal vector (and thereby the downstream properties) by employing the shock wave relations which link the flow turning angle and the shock wave angle, both these angles being measured from the approach streamline direction in a plane defined by the approach velocity vector and the shock wave normal vector. Since the shock wave normal vector is required to define this plane, an iterative procedure for determining that vector is required in this method. This method was tested and produced results identical to the method described earlier. However, due to the greater complexity of the alternate method, it was not selected for use in the final algorithm.

8. SHOCK-MODIFIED INTERIOR POINT UNIT PROCESSES

In some situations during the computation of the internal flow, the interior point unit process must be applied in a modified form. One such application is illustrated in Figure E.6. In this situation, the Mach cone, with apex at the solution point, intersects not only the initial-value plane but also a solid boundary and an internal shock wave. The point notation used in Figure E.6 is the same as that used in the computational network of the basic interior point scheme, which is illustrated in Figure E.1. The solution point, denoted by point (6)

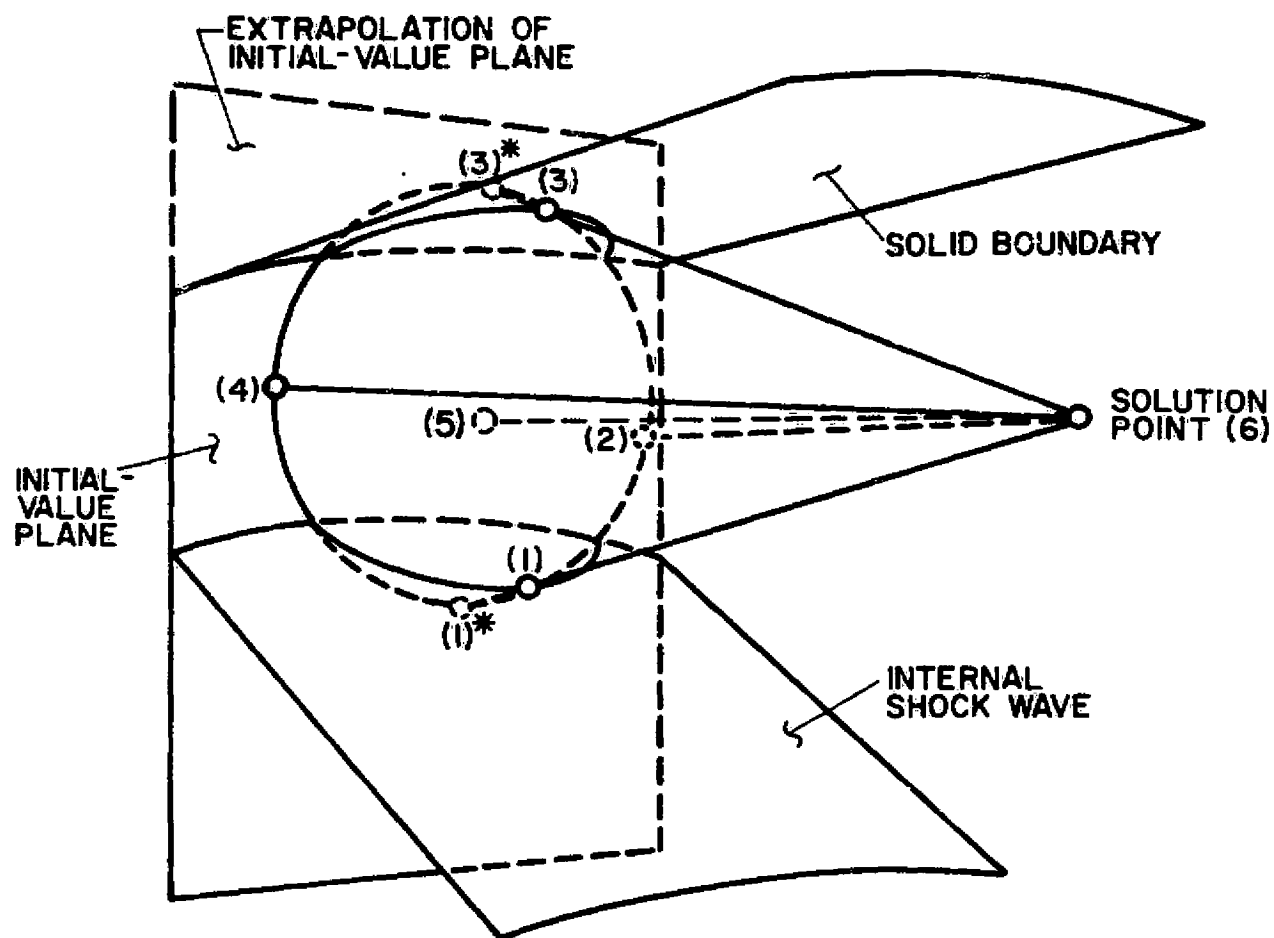


FIGURE E.6. SHOCK-MODIFIED INTERIOR POINT COMPUTATIONAL NETWORK (STREAMLINE BASE POINT ON INITIAL-VALUE PLANE)

in Figure E.6, lies on the current solution plane. Point (5) represents the streamline base point on the initial-value plane. As in the basic interior point unit process, points (1) to (4) represent the bicharacteristic base points. Point (1), in this case, lies on the surface of the internal shock wave, and point (3) lies on the solid boundary. Points (2) and (4) lie on the initial-value plane.

The axial distance between the initial-value plane and the solution plane is determined by either the CFL stability criterion or by the special constraints which apply when an internal shock wave intersects a solid boundary. Those procedures are discussed in Appendix F. In either case, the axial step is determined prior to the application of the unit processes.

In the overall algorithm for the computation of the internal flow, the order of integration is selected so that the shock wave solution points and the body solution points are determined before any attempt is made to obtain the solution at any of the interior field points which lie in the flow field sector that is downstream of the shock wave. As a consequence, the flow property fields on the downstream side of the shock wave and on the stream surface formed by the solid boundary are determined before the solution at an interior point, such as point (6) in Figure E.6, is attempted.

The procedure used to obtain the solution at point (6) in Figure E.6 is almost identical to the basic interior point unit process, which is presented in Section 4 of this appendix. The major difference between the two algorithms is that, in the present case, the bicharacteristic intersection points on the shock wave [point (1)] and on the

solid boundary [point (3)] must be determined in addition to those bi-characteristic intersection points [points (2) and (4)] on the initial-value plane. Along with the location of these points, flow property values and first partial derivatives of the flow properties at these points must also be obtained.

As in the basic interior point unit process, flow property values at points (2) and (4) on the initial-value plane are obtained using the bivariate interpolation polynomial given by equation (E.11). The coefficients in this equation are determined by a least squares fit of nine data points in the initial-value plane as discussed in Appendix C. Flow property values at point (1) on the shock wave surface or at point (3) on the solid boundary surface are obtained using the tri-variate interpolation polynomial

$$f(x,y,z) = a_1 + a_2y + a_3z + a_4yz + a_5y^2 + a_6z^2 + a_7xy + a_8xz \quad (E.67)$$

The coefficients a_i ($i=1$ to 8) in equation (E.67) are determined by a least squares fit of fourteen data points on either the downstream side of the shock wave for interpolation on that surface, or on the solid boundary for interpolation on that surface. The detailed implementation of equation (E.67) is presented in Appendix C.

An outline of the unit process used to determine the solution at point (6) in Figure E.6 is now presented. The computation is initiated by determining the location of the solution point, point (6), using equation (E.12) in a manner identical to the procedure employed in the basic interior point unit process. After the position of the solution

point has been obtained for a given outer iteration, the four bi-characteristics, corresponding to the values of the parametric angle $\theta = 0, \pi/2, \pi$, and $3\pi/2$ in equation (E.13), are extended rearward from the solution point to the initial-value plane. From the bicharacteristic-initial-value plane intersection point coordinates, denoted by $y^*(k)$ and $z^*(k)$ ($k=1$ to 4), the radius $r^*(k) = [y^*(k)^2 + z^*(k)^2]^{1/2}$ and the polar angle $\theta^*(k) = \tan^{-1}[z^*(k)/y^*(k)]$ of each intersection point are computed. The radius $r^*(k)$ is then compared to the shock wave radius r_s and the body radius r_b in the meridional plane defined by the polar angle $\theta^*(k)$. The shock wave radius is determined from the univariate interpolation polynomial

$$r_s(\theta) = a_1 + a_2\theta + a_3\theta^2 \quad (\text{E.68})$$

where the coefficients a_i ($i=1,2,3$) are determined by fitting this expression to three shock wave solution points in the initial-value plane as described in Appendix C. The solid body radius r_b is obtained by employing the formulations presented in Appendix D. For the orientation shown in Figure E.6, if $r_s \leq r^*(k) \leq r_b$, the bicharacteristic intersects the initial-value plane and the analysis proceeds as in the basic interior point unit process. If $r^*(k) < r_s$, the bicharacteristic intersects the internal shock wave. In this case, the bicharacteristic base point location on the surface of the shock wave is found by employing the bicharacteristic-surface intersection scheme presented in Appendix D. For a shock wave intersection, that scheme requires that equation (E.68) also be fitted to three shock wave solution points in the current solution plane. If $r^*(k) > r_b$, the bicharacteristic

intersects the solid boundary. The bicharacteristic base point location on the solid boundary is also obtained by using the iterative scheme presented in Appendix D. As in the basic interior point unit process, an inner iteration is performed for locating points (1) to (4). Interpolated values of the flow properties at the respective points are obtained by using either equation (E.11) or equation (E.67), whichever is applicable.

After the bicharacteristic base points, points (1) to (4), have been located, the first partial derivatives of the flow properties with respect to y and z at these points are obtained by analytically differentiating the appropriate interpolation polynomial. In a like manner, these derivatives are also obtained at the streamline base point, point (5). Then, using the governing partial differential equations, the x -partial derivatives of the flow properties are found at points (1) to (5). For any bicharacteristic which intersects the shock wave or the solid boundary, the time parameter $[t(6) - t(k)]$ is found using equation (E.13) applied for $i=1$ (i.e., the x -coordinate direction) while employing the appropriate intersection coordinates. At this stage, the system of compatibility equations may be solved for the flow properties at point (6) in a manner identical to that employed in the basic interior point scheme.

The situation illustrated in Figure E.6 is quite general. In some instances, there are no bicharacteristic intersections with the solid boundary. Alternatively, there may be no intersections of the bicharacteristics with the internal shock wave. There may be two bicharacteristics intersecting with the shock wave, etc.

Another situation in which the interior point unit process must be applied in a modified form is illustrated in Figure E.7. In this figure, the Mach cone, with apex at the solution point, intersects both the initial-value plane and the internal shock wave. The point notation used in Figure E.7 is the same as that used in Figure E.6. However, in this case, the streamline base point, point (5), does not lie on the initial-value plane, but rather lies on the surface of the internal shock wave.

The location of the streamline base point is obtained by extending the streamline from the initial-value plane to the surface of the shock wave. The point of intersection of the streamline with the shock wave is determined by employing the iterative scheme which is presented in Appendix D for finding a streamline-surface intersection point. That procedure requires that equation (E.68) be applied to three known shock wave solution points in the initial-value plane and three shock wave solution points in the current solution plane. Furthermore, interpolated values of the velocity components are required on the upstream side of the shock wave at the point where the streamline intersects the shock wave. For this purpose, the following linear tri-variate interpolation polynomial is employed.

$$f(x,y,z) = a_1 + a_2x + a_3y + a_4z \quad (\text{E.69})$$

The coefficients a_i ($i=1$ to 4) in equation (E.69) are determined by fitting this expression to four data points on the upstream side of the shock wave, as discussed in Appendix C.

After the streamline-shock wave intersection point has been determined, the following fraction is formed

$$\epsilon = [x_S - x(5)] / (x_S - x_I) \quad (E.70)$$

where x_I and x_S are the axial positions of the initial-value plane and the solution plane, respectively. If ϵ is greater than a specified minimum value, an interior point unit process is performed on the downstream side of the shock wave. This unit process is almost identical to that used for determining the solution at point (6) in Figure E.6. In this case, however, the streamline formula given by equation (E.12) is applied between the streamline-shock wave intersection point and the solution plane. Interpolated flow property values at point (5) are determined by applying equation (E.67) to fourteen data points on the downstream side of the shock wave.

If, on the other hand, ϵ is less than the specified minimum value, an interior point unit process on the downstream side of the shock wave is not performed. Instead, the streamline from point (5) is projected onto the solution plane, and the flow properties at the solution point are determined by interpolation in the solution plane. The streamline integration from point (5) to point (6) employs equation (E.12). The flow property values at point (5) are obtained from equation (E.67) applied to fourteen data points on the downstream side of the shock wave. Flow property values at the streamline-solution plane intersection point are determined from the linear bivariate polynomial

$$f(y,z) = a_1 + a_2 y + a_3 z \quad (E.71)$$

The coefficients a_i ($i=1,2,3$) in equation (E.71) are determined by fitting this expression to three data points in the current solution plane, as described in Appendix C. The order of integration for

determining the internal flow field is specified so that the downstream shock wave points and outer interior points in the downstream flow field sector are determined first. The location of the solution point, in this case, is determined by an iterative loop which is terminated when the y and z coordinates of the projected solution point have converged.

9. SHOCK-MODIFIED SOLID BOUNDARY POINT UNIT PROCESSES

In some situations, the solid boundary point unit process must be applied in a modified form. One such application is illustrated in Figure E.8. In this situation, a portion of the Mach cone, with apex at the solid body solution point, intersects both the initial-value plane and the internal shock wave. The point notation used in Figure E.8 is identical to that used in Figure E.2, which depicts the computational network for the standard body point unit process. The unit process employed in the present case is almost identical to the standard body point unit process. In the present case, however, the bicharacteristic-shock wave intersection is handled in a manner identical to that employed in the shock-modified interior point unit process presented in the previous section.

In some situations, the entire Mach cone intersects the shock wave, as illustrated in Figure E.9. This situation occurs at a body point on the solution plane that is immediately downstream of a solid body-shock wave reflection, or at a body point on the solution plane that is immediately behind the shock wave emanating from the cowl lip. In the former case, the shock wave-solid body intersection is a space curve in three-dimensions, whereas, in the latter case, the shock wave-solid

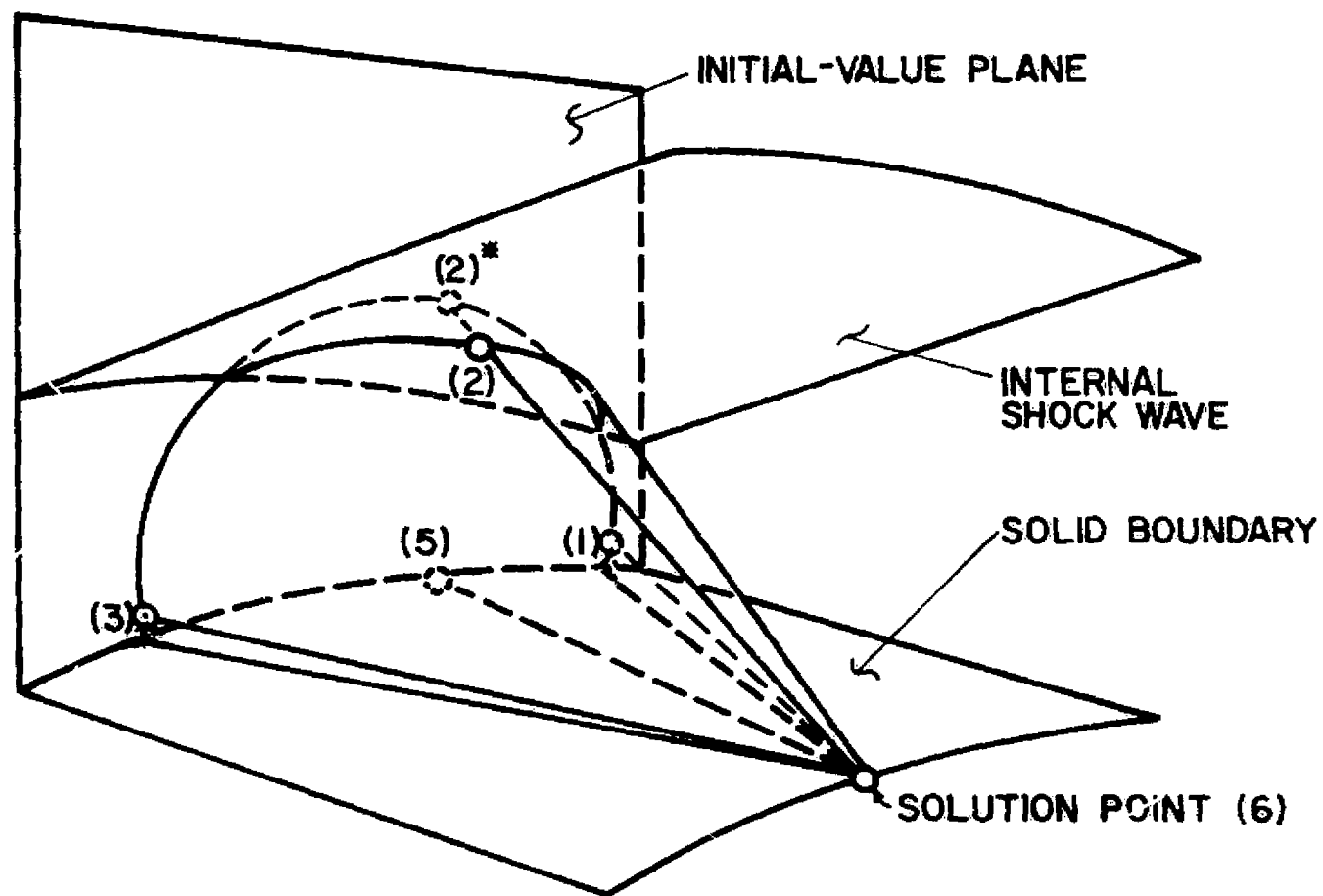


FIGURE E.8. SHOCK-MODIFIED SOLID BOUNDARY POINT COMPUTATIONAL NETWORK (TYPICAL APPLICATION)

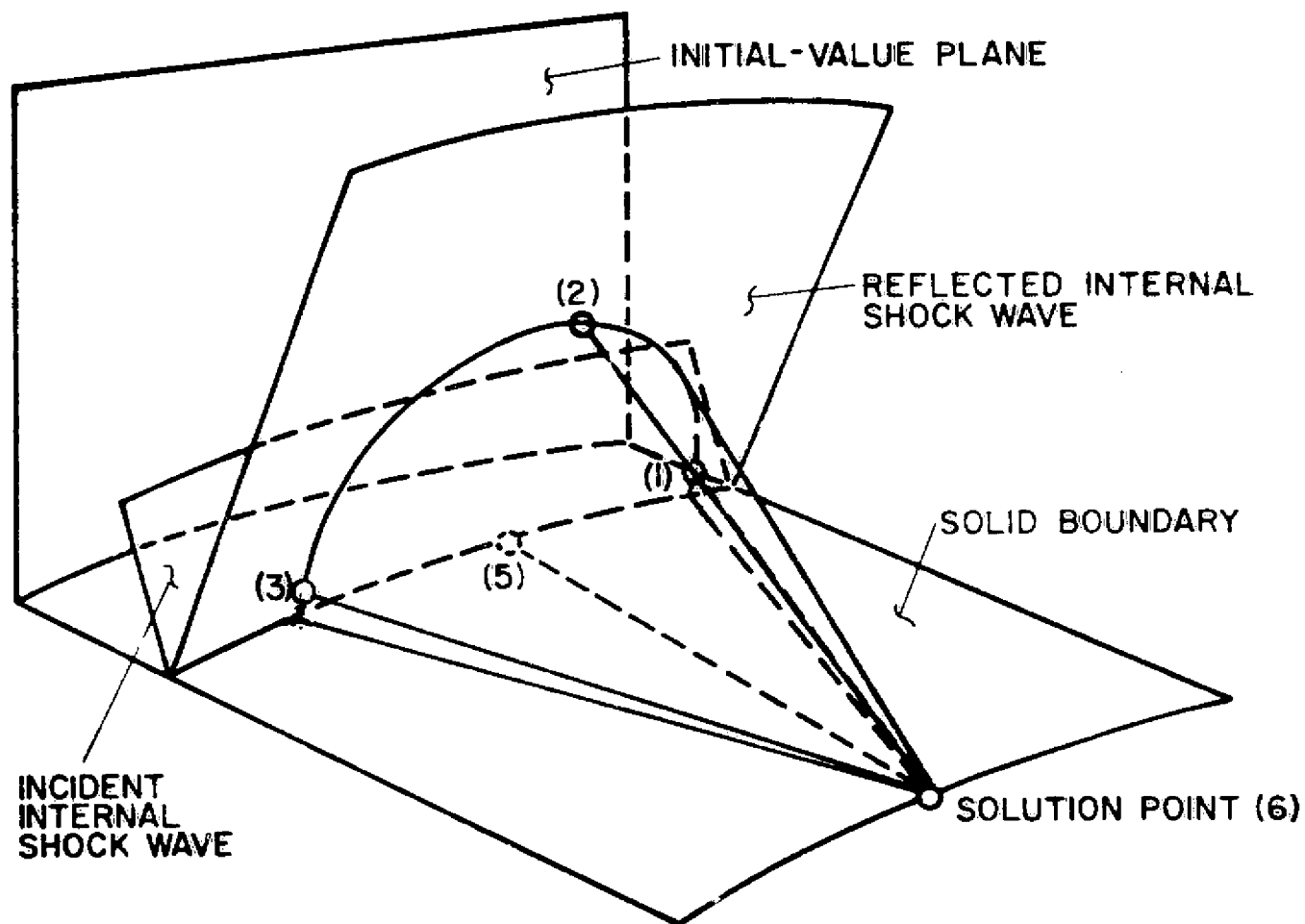


FIGURE E.9. SHOCK-MODIFIED SOLID BOUNDARY POINT COMPUTATIONAL NETWORK (POST SHOCK REFLECTION APPLICATION)

body intersection is a curve in a plane of constant x . The appropriate intersection algorithm is used as presented in Appendix D, and for the most part, procedures identical to those employed in the shock-modified interior point unit process are employed in this case.

10. INTERNAL FLOW FIELD-SHOCK WAVE POINT UNIT PROCESSES

Figure E.10 illustrates the overall computational network used in determining the solution for a typical shock wave point in the internal flow field. To determine the solution at the shock wave point, an interior point unit process must be performed to obtain the upstream flow properties at the location of the shock wave solution point. Figure E.10 illustrates both the computational network for the interior point unit process (denoted by primed numbers), and the computational network for the standard shock wave point unit process (denoted by unprimed numbers). The point notations employed in these computational networks are identical to those used in the corresponding standard unit processes.

The computational procedure employed for determining the solution for an internal flow field-shock wave point is almost identical to the bow shock wave point unit process. The major difference between the two procedures is that for an internal flow shock wave point, the upstream flow properties at the solution point are obtained from an interior point computation, rather than using free-stream data as in the bow shock wave point unit process. The required interior point unit process is essentially the same as the basic interior point unit process presented in Section 4 of this appendix. In the present case, however, the streamline is not extended from a field point in the

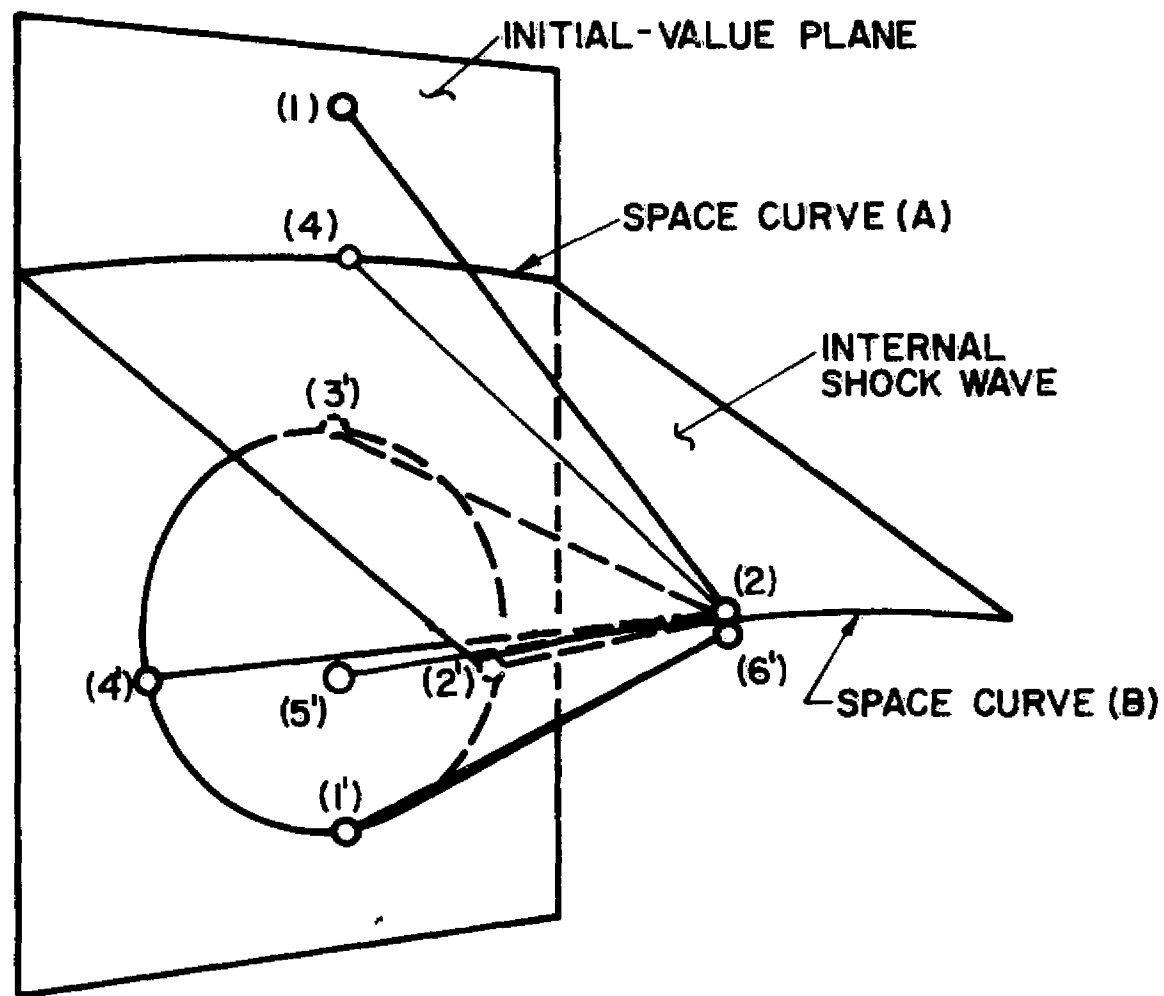


FIGURE E.10. INTERNAL FLOW FIELD-SHOCK WAVE POINT COMPUTATIONAL NETWORK (TYPICAL APPLICATION)

initial-value plane to the solution plane, but rather it is extended from the shock wave solution point back to the initial-value plane. The position of the shock wave solution point is determined by the shock wave point unit process. To initiate the interior point computation in the present case, flow property values are used from an adjacent field point in the flow field sector that is upstream of the shock wave in the solution plane. This modified interior point unit process requires searching the flow field sector upstream of the shock wave in the initial-value plane for the field point that is closest to the streamline-initial-value plane intersection point. This point is then used as the base point for the stencil of initial-value plane field points that are used in formulating the bivariate interpolation polynomial given by equation (E.11) (see Appendix C).

For the first solution plane inside the inlet, the downstream bicharacteristic base point, point (1) in Figure E.11, does not lie on the initial-value plane, but rather is located on the stream surface formed by the cowl boundary. To compute the pressure at point (2) from the wave surface compatibility relation, equation (E.59), the flow property values must be available at point (1), which requires that the flow property field must be known on the cowl surface. The body points on the cowl surface at the first internal flow solution plane, however, must be obtained from the unit process described in Section 9 of this appendix. That unit process requires that the flow property field on the downstream side of the shock wave be known. Hence, a simultaneous solid body point-shock wave point algorithm must be employed. This procedure was not developed in the present

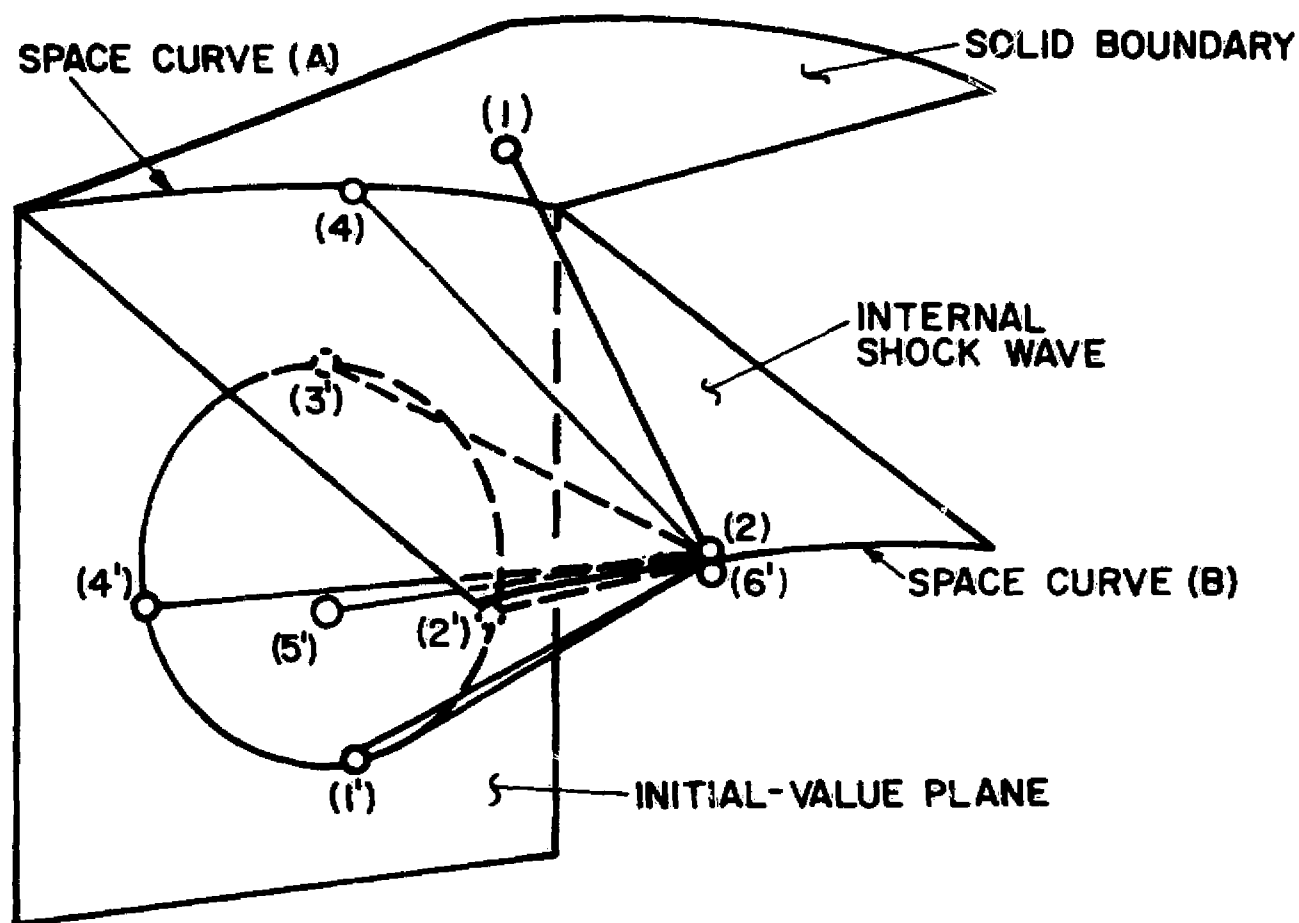


FIGURE E.II. INTERNAL FLOW FIELD-SHOCK WAVE POINT COMPUTATIONAL NETWORK (FIRST INTERNAL FLOW SOLUTION PLANE APPLICATION)

investigation. Rather, the shock wave points on the first internal flow solution plane are computed using a value of ϕ in equation (E.37) equal to the value of ϕ at the shock wave point in the initial-value plane which lies in the same meridional plane as the solution point. This provides a solution at each shock wave point on the first solution plane without employing the compatibility relation along the bicharacteristic. The body points on the cowl are then computed in the manner outlined in Section 9. On ensuing solution planes, except for the one immediately after a solid body-shock wave intersection, the bicharacteristic base point is located and the angle ϕ is iterated to convergence.

When the internal shock wave intersects a solid boundary, as illustrated in Figure E.12, a modification is required to the shock wave point unit process. In this case, instead of performing an interior point unit process to obtain the upstream flow properties at the solution point, a modified solid boundary point unit process must be employed. Moreover, the shock wave solution point, in this case, does not lie on the solution plane, but rather its position must be obtained by computing the intersection of the incident shock wave with the solid boundary.

Finally, it should be noted that in order to achieve strict second-order accuracy in the internal flow shock wave point solution, global correction must be performed [this involves evaluating the cross derivatives at the solution point and using updated values of α in equation (E.38)]. Time constraints in the present investigation did not permit the development of the global correction capability for

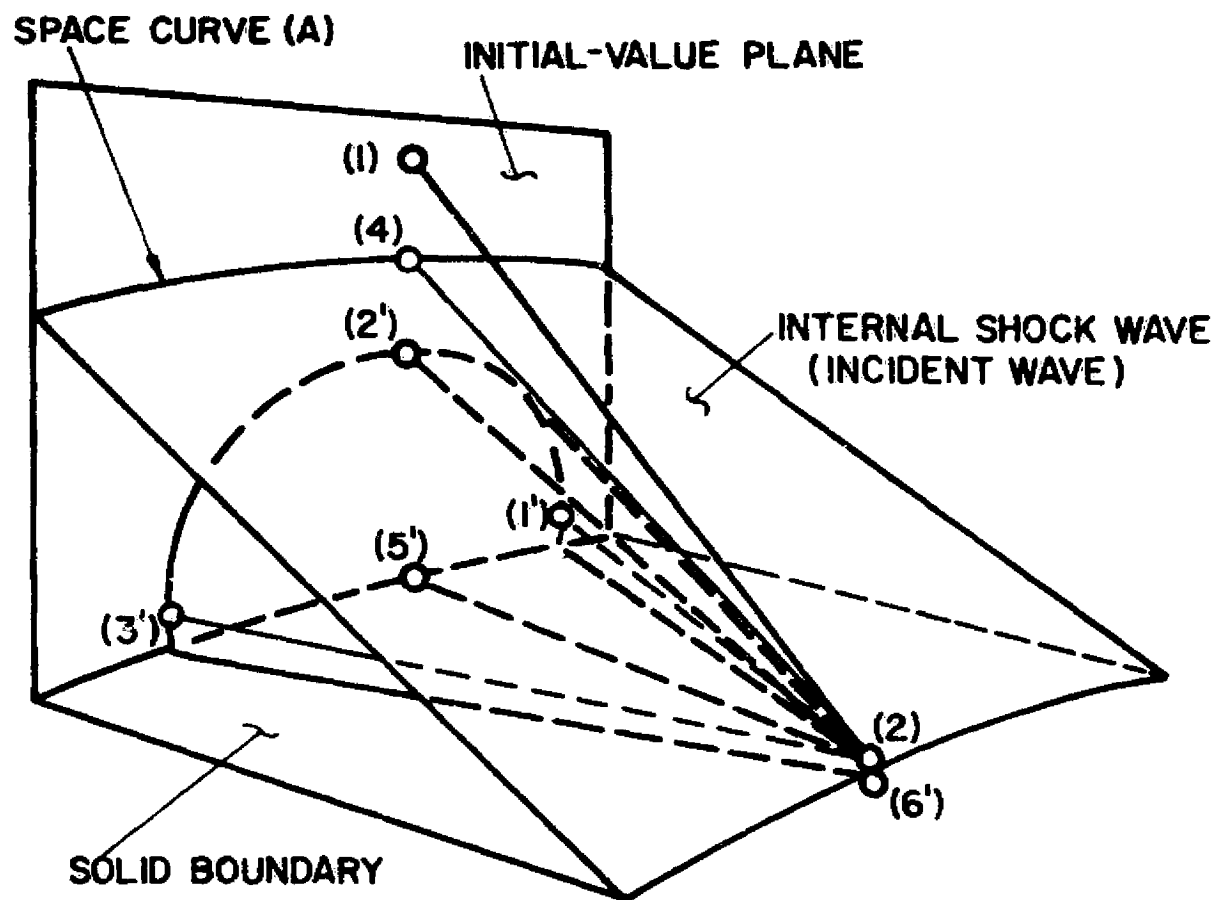


FIGURE E.12. INTERNAL FLOW FIELD-SHOCK WAVE POINT COMPUTATIONAL NETWORK (INCIDENT WAVE-BOUNDARY INTERSECTION APPLICATION)

the internal flow shock wave points. Hence, only local iteration can be performed for those points.

APPENDIX F

OVERALL NUMERICAL ALGORITHM

1. INTRODUCTION

The overall numerical algorithm consists of the repetitive application of the various unit processes to generate the global solution for given boundary conditions and a specified set of initial data.

The boundary conditions are represented by the formulations presented in Appendix D. The initial data are specified on a space-like plane of constant x . The x -coordinate axis is the longitudinal axis of the centerbody and the cowl. Moreover, the mean flow direction is assumed to be in the x -coordinate direction.

An inverse marching scheme is employed in the overall numerical algorithm. The solution is obtained on space-like planes of constant x . The solution points on each plane represent the intersection points of continuous streamlines which are propagated from the data points specified on the initial plane. In addition to the streamline solution points, are the solution points representing the intersection of either the external or the internal shock wave with the solution plane. For the internal flow, the solution is also obtained on the space curves which represent the intersection of the internal shock wave with the solid boundaries. These space curves are defined by the locus of shock wave solution points.

Except in the vicinity of a shock wave reflection with a solid boundary, the axial (x) distance between successive solution planes is determined by the application of the Courant-Friedrichs-Lewy (CFL) stability criterion. In the vicinity of a shock wave intersection with a solid boundary, the axial step is controlled by special constraints which insure that the entire shock wave-solid boundary intersection falls between two adjacent solution planes.

After each solution plane is computed, the mass flow rate across that plane is calculated using trapezoidal rule integration. Constancy of the overall mass flow rate in the internal flow field computation gives an indication of the overall accuracy of the numerical integration. The stagnation pressure and stagnation temperature are calculated at each solution point. For the adiabatic flow of a calorically perfect gas, the stagnation temperature should remain constant.

In the numerical analysis, the flow field is divided into two regimes: the internal flow regime and the external flow regime, as illustrated in Figure F.1. The flow field integration in each of these two regimes is controlled by separate logic modules in the computer program. The forebody flow field integration is performed first. Then, the internal flow field is computed. The computer program developed in the present investigation has the capability to perform the internal flow field integration with or without the discrete fitting of the internal shock wave system. The option in which shock waves are not discretely fitted might be employed if the internal shock waves are of relatively weak strength, and thereby an acceptable solution could be obtained by smearing the internal discontinuities.

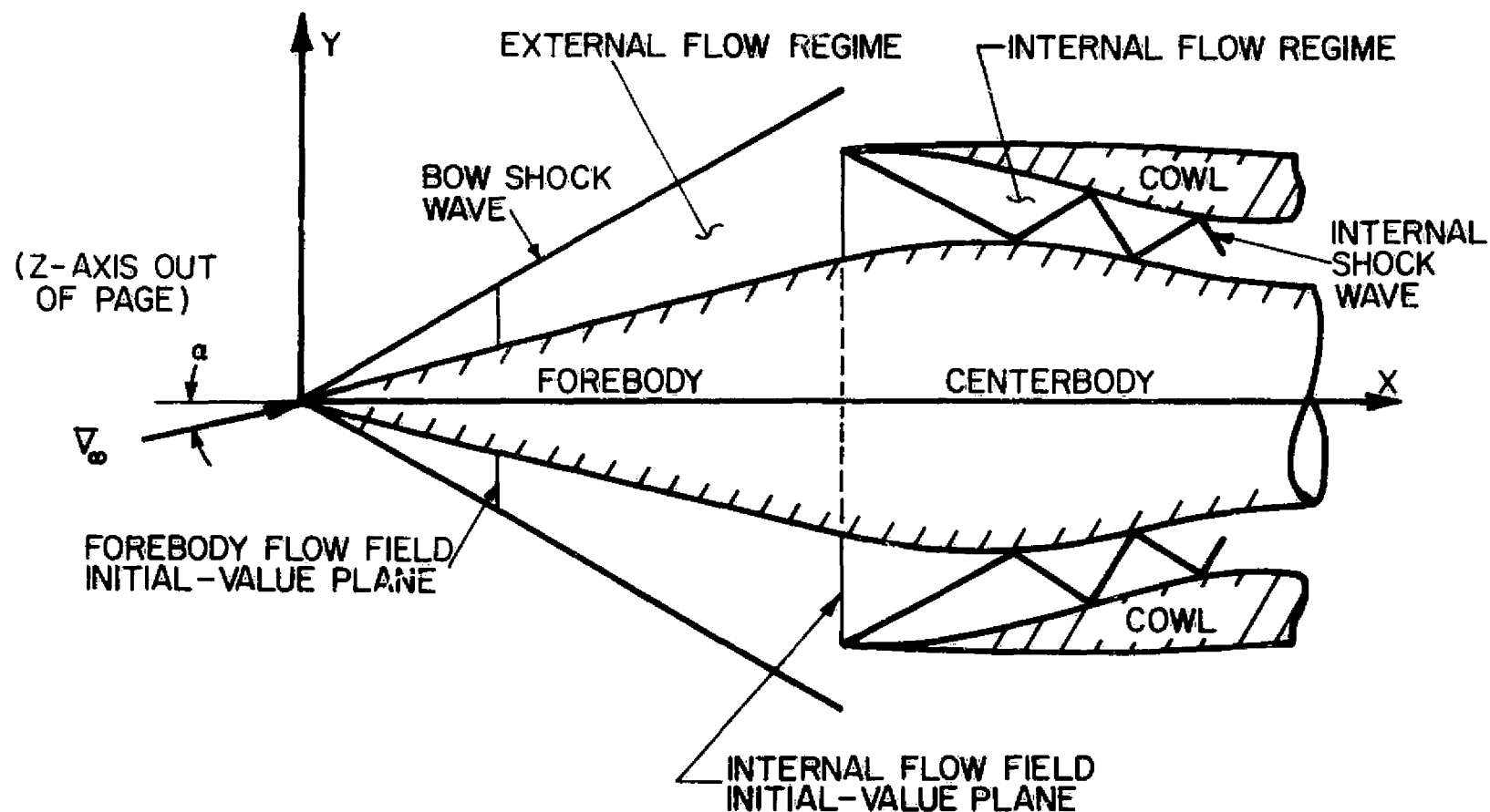


FIGURE F.1. MIXED-COMPRESSION AIRCRAFT INLET

From a computation point of view, the internal flow field in which shock waves are not discretely fitted is the easiest solution to compute. For flow fields in which shock waves are discretely fitted, the external flow about the forebody is less difficult to obtain than the internal flow, since in the external flow the shock wave represents a bound to the computational regime. Discrete fitting of the shock wave throughout the computational regime, as is done in the internal flow field integration, greatly complicates the numerical algorithm.

In this appendix, the overall control logic used in each of the three flow field integration options is discussed. Regulation of the axial marching step size, generation of the initial-value surface data, and considerations of flow symmetry are also discussed. All of the unit processes referred to in this appendix are discussed in Appendix E.

2. COURANT-FRIEDRICHS-LEWY (CFL) STABILITY CRITERION

Except in the vicinity of an internal shock wave-solid boundary intersection, the axial marching step between successive solution planes is determined by the application of the Courant-Friedrichs-Lewy (CFL) stability criterion (9). The CFL stability criterion will be satisfied at each solution point if the convex hull of the finite difference network contains the differential zone of dependence of the solution point. The convex hull of the finite difference network, illustrated in Figure F.2, is defined by the outer periphery of initial-value plane field points used in determining the fit point stencil for the quadratic bivariate interpolation polynomial. The

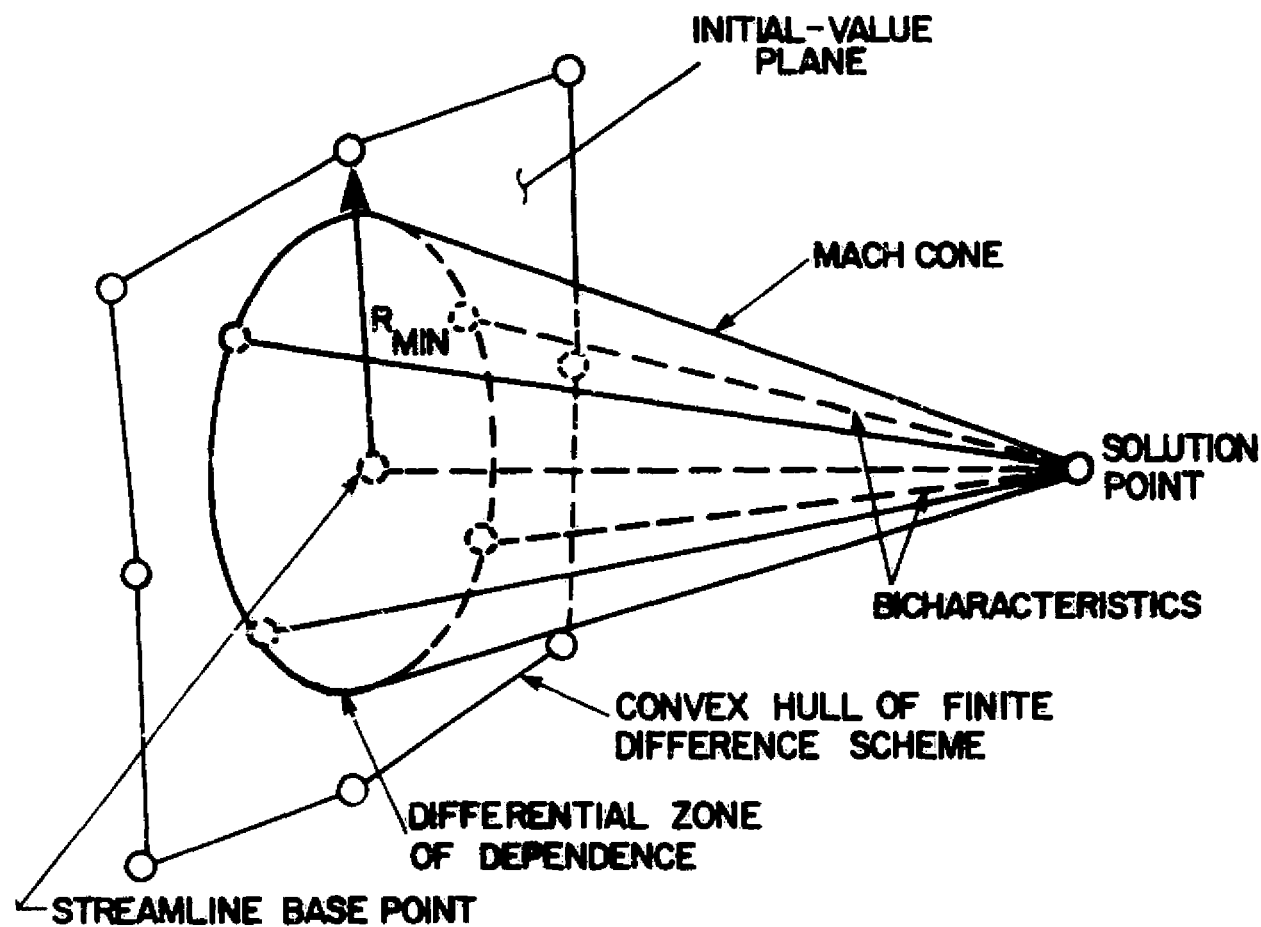


FIGURE F.2. CFL STABILITY CRITERION

differential zone of dependence, also illustrated in Figure F.2, is the region defined by the intersection of the Mach cone (whose apex is at the solution point) with the initial-value plane.

The maximum allowable marching step for each streamline is the x-step for which the Mach cone just touches the convex hull. That step size is given by

$$\Delta x = [u^2/(cq)][1 - (c/q)(q^2/u^2 - 1)^{1/2}]R_{\min} \quad (F.1)$$

where Δx is the maximum allowable axial step, u is the x-component of the velocity, q is the velocity magnitude, and c is given by

$$c^2 = a^2 q^2 / (q^2 - a^2) \quad (F.2)$$

where a is the local sonic speed. In equation (F.1), $R_{\min}$ is the distance from the streamline base point in the initial-value plane to the nearest field point on the convex hull of the finite difference network (see Figure F.2).

Equation (F.1) is applied at every streamline solution point, the actual marching step being selected as the Δx value at the most restrictive point. It should be noted that this expression is applied only to streamline points, the shock wave points being excluded. Furthermore, in the internal flow field integration, the shock wave points are ignored in defining the convex hull of the finite difference network when application of the stability criterion is made to a streamline point.

3. INITIAL-VALUE PLANE

The initial data are specified on a plane of constant x . The flow must be supersonic at every point on this plane. For uniqueness and existence of a genuine solution, the values of the five dependent variables (u , v , w , P , and ρ) prescribed on this surface must have at least continuous first derivatives.

If the forebody flow field is to be determined, the initial-value plane must be specified at an axial (x) station that is upstream of the forebody computational flow regime (see Figure F.1). The solution is then found along the streamlines that pass through the data points specified on the initial-value plane, although some streamline addition and deletion are performed on the ensuing solution planes as described in Section 5 of this appendix.

If only the internal flow field is to be determined, the initial-value plane must be specified at the axial station which corresponds to the x -position of the cowl lip (see Figure F.1). The cowl lip is assumed to be contained in a plane of constant x . For the integration of the internal flow field, a point redistribution is performed on the initial-value plane. This point redistribution is required in order to have streamlines which lie in the stream surface formed by the cowl boundary. The solution is then found along the streamlines that pass through the redistributed points on the plane at the cowl lip axial station.

The initial-value plane may be specified by the user, or if the forebody is conical up to the axial station where the initial-value plane is located, the flow property field on the initial-value plane

can be generated internally in the computer program. The internally generated initial-value plane is obtained by an approximate technique which employs the Taylor-Maccoll solution for the flow about a circular cone at zero incidence. A superposition procedure is used to obtain an approximation to the flow about a circular cone at nonzero angle of attack by neglecting the cross flow effects. This superposition procedure effectively amounts to computing the flow turning angle in the meridional plane of the given solution point, and then obtaining the flow properties at that point by applying the Taylor-Maccoll solution for a cone half-angle equal to the flow turning angle. The shock wave angle is then measured from the original streamline direction in the appropriate meridional plane. It must be emphasized that this is only an approximate technique, giving the well accepted Taylor-Maccoll solution at zero incidence, but becoming increasingly less accurate as the angle of attack is increased.

The solution obtained by Jones (33) for the flow about a circular cone at nonzero incidence has been well substantiated. Using a conversion algorithm, the results of the computer program developed by Jones can be made compatible with the input data required by the computer program developed in the present investigation. Many of the computed results presented in Section VI were obtained using the results of Jones' program as initial data. For situations in which the forebody is conical up to the axial station where the initial-value plane is located, the Jones program is the recommended source for the initial data.

If the forebody is not conical ahead of the axial station of the initial-value plane, another source of initial data must be used. If available, experimental data may be employed.

4. SOLUTION PLANE POINT NETWORK AND FLOW SYMMETRY

The computational point network is based on a series of circumferential and radial stations. The point networks for the various flow symmetry options are illustrated in Figure F.3. In this figure, the index i corresponds to the i th circumferential station and the index j corresponds to the j th radial station. In all cases, the streamlines on the surface of the centerbody are denoted by $j = 1$. For the forebody flow field, the bow shock wave solution points are denoted by $j = n$. For the internal flow field, the streamlines on the surface of the cowl are denoted by $j = n$. The computed sector, in general, is bounded by the circumferential stations corresponding to $i = 1$ and $i = m$. This point arrangement produces a rectangular logic array in the computer program.

The points at any circumferential station in axisymmetric flow, or on a plane of flow symmetry in three-dimensional flow, lie on a straight line. Moreover, for axisymmetric flow, the radial stations correspond to circular rings. In general, however, the solution points at a given circumferential station do not lie on a ray, nor do the radial stations correspond to circular rings.

For the internal flow option in which shock waves are discretely fitted, the shock wave solution points are also represented in this point arrangement. Special logic is used in the computer program such that the shock wave solution points float in the storage arrays

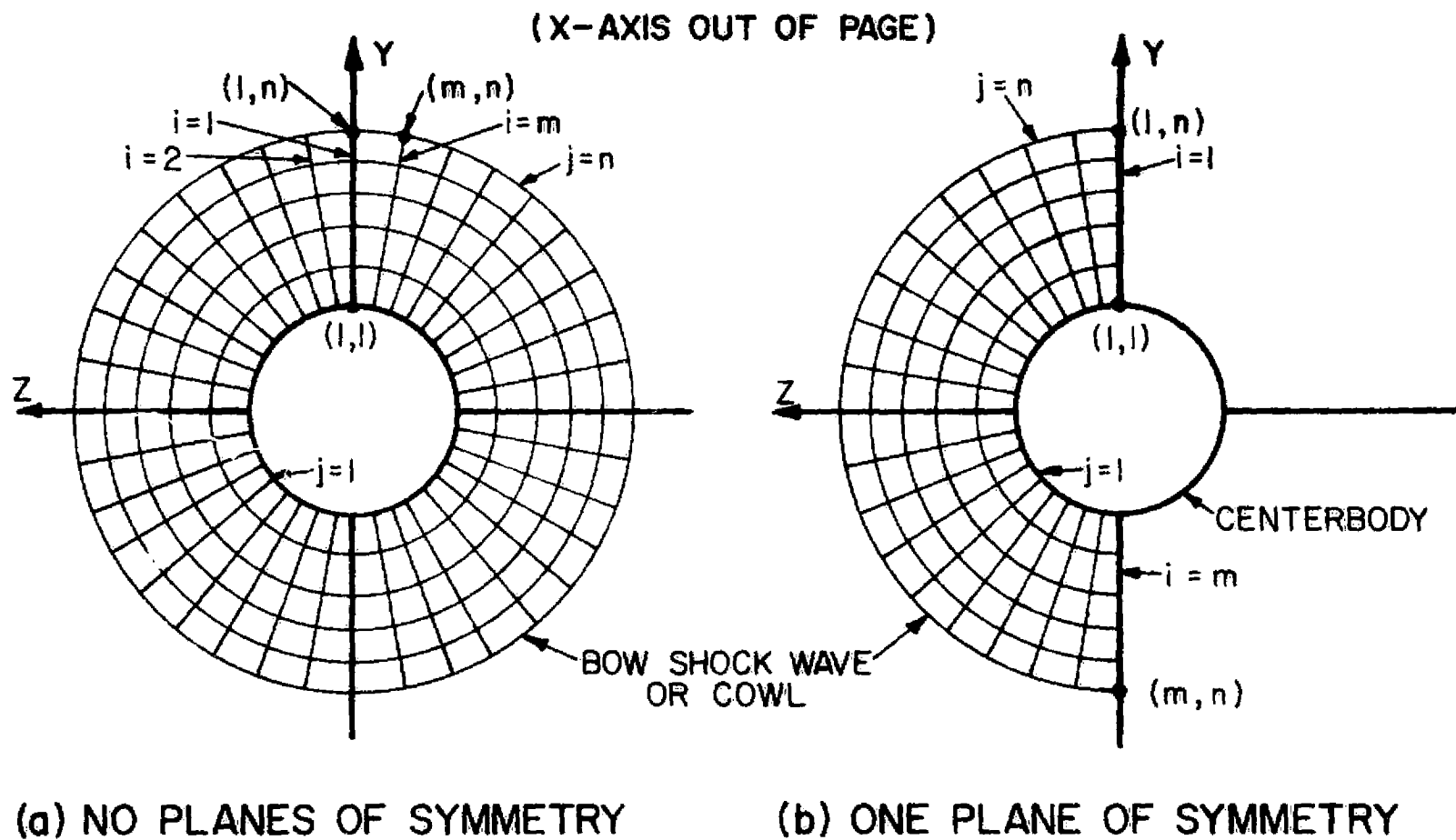
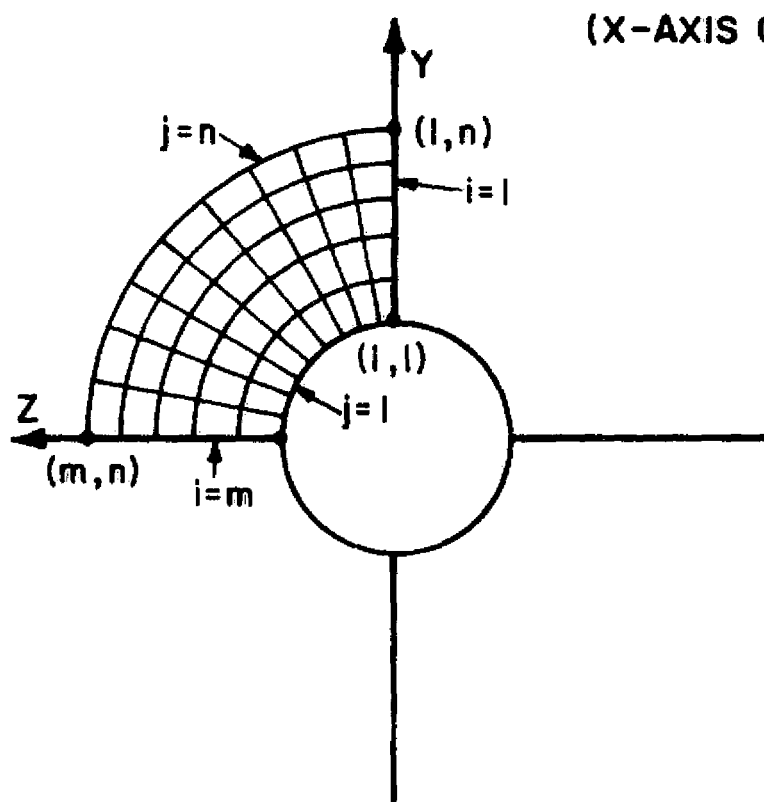
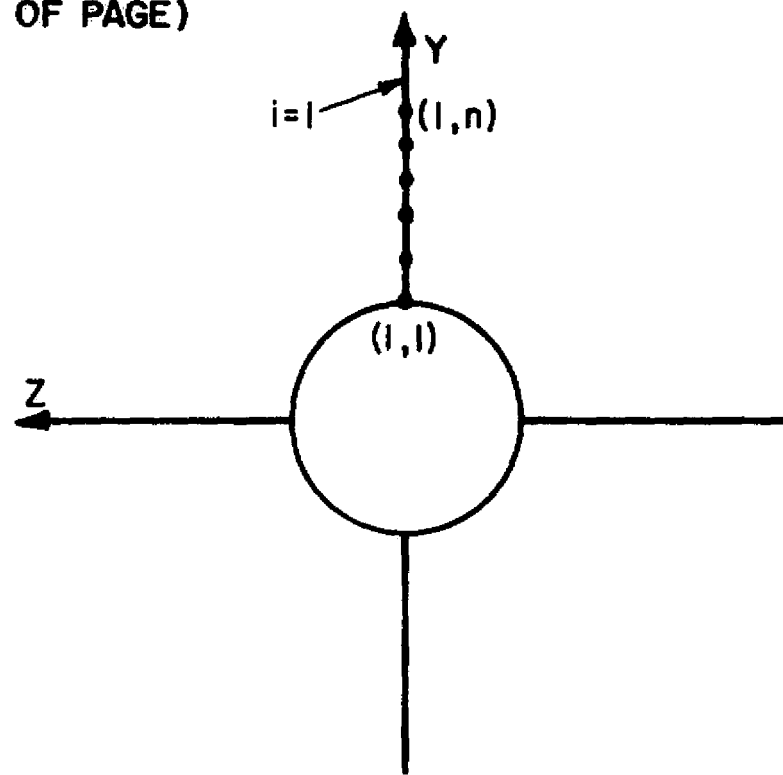


FIGURE F.3. COMPUTATIONAL POINT NETWORKS



(c) TWO PLANES OF SYMMETRY



(d) AXISYMMETRIC FLOW

FIGURE F.3. (CONTINUED)

as the shock wave travels between the centerbody and cowl on successive solution planes. On a given solution plane, the shock wave solution points at adjacent circumferential stations do not, in general, have to lie at the same radial station.

The computer program takes advantage of flow symmetry when it exists in the flow field. In these instances, the entire solution plane does not have to be computed, but rather only an appropriate section of it. The remaining sections of the solution plane may be obtained by reflection of the points in the computed sector. This procedure yields a significant reduction in computer execution time.

The four flow symmetry options that have been incorporated into the analysis are depicted in Figure F.3. Figure F.3(a) illustrates the most general case when no flow symmetry is present. Figure F.3(b) illustrates the case when one plane of flow symmetry is present. In this case the computed sector is the half-plane bounded by the y -axis and containing the $+z$ -axis. The integration region in this case is bounded by the $i = 1$ circumferential station on the $+y$ -axis and by the $i = m$ circumferential station on the $-y$ -axis. This case of flow symmetry is the one most likely to arise in the class of problems being considered in this investigation. Figure F.3(c) illustrates the case when two planes of flow symmetry are present. This option would be used to compute the flow field about asymmetric bodies at zero angle of attack. In this instance, the computed sector is the quadrant bounded by the $+y$ -axis and the $+z$ -axis. The circumferential station corresponding to $i = 1$ lies on the $+y$ -axis and the circumferential station corresponding to $i = m$ lies on the

+z-axis. Finally, Figure F.3(d) illustrates the axisymmetric flow option where the computed sector is limited to the single circumferential station (ray) lying on the +y-axis. This option would be used to compute the flow field about axisymmetric bodies at zero angle of attack.

The numerical algorithm does not apply special unit processes when a solution point lies on a plane of symmetry. Rather, a point reflection about the plane of symmetry is performed in the initial-value plane, and the appropriate unit process is then applied in standard form. This procedure yields satisfactory results and eliminates the need for devising special unit processes.

5. EXTERNAL FLOW ABOUT THE FOREBODY

With the forebody geometry specified and the flow property field on the initial-value plane determined, the external flow about the forebody can be calculated. In the computation of this flow field, the distance between successive solution planes is determined by the application of the CFL stability criterion. The last solution plane in the forebody flow field computation is made to coincide with the x-position of the cowl lip.

After the axial step between the current initial-value plane and the current solution plane has been determined, the solid boundary point unit process (see Appendix E) and the interior point unit process (see Appendix E) are applied. These unit processes achieve second-order accuracy without the need for global iteration. Hence, these unit processes are applied at the appropriate points until convergence is obtained without using information from neighboring points

in the solution plane.

Once the solution at each solid boundary point and interior point has been determined, the bow shock wave point unit process (see Appendix E) is applied at each shock wave solution point in the computed sector. Global correction is then applied for these points, if desired. The position of each shock wave solution point is made to lie in the meridional plane defined by the outer-most interior field point which is on the same circumferential station as the shock wave point. As a consequence, in axisymmetric flow, the streamline and shock wave solution points on a given circumferential station lie in the same meridional plane on all succeeding solution planes. In three-dimensional flow, however, except on a plane of flow symmetry, the solution points corresponding to a given circumferential station do not lie in the same meridional plane on successive solution planes.

In the forebody flow field integration, periodic streamline addition and deletion are performed. The streamline addition is required to retain a well-dispersed computational mesh, since at successive solution planes more and more mass is captured. Moreover, convergence of the streamlines towards the forebody occurs as the flow progresses downstream. Periodic point deletion is required since the continued addition of streamlines would produce an excessively large number of computational mesh points, thereby unduly increasing computer execution time and machine storage requirements. The streamline addition and deletion procedures are outlined in the following. A depiction of a typical forebody flow streamline pattern is given in Figure F.4.

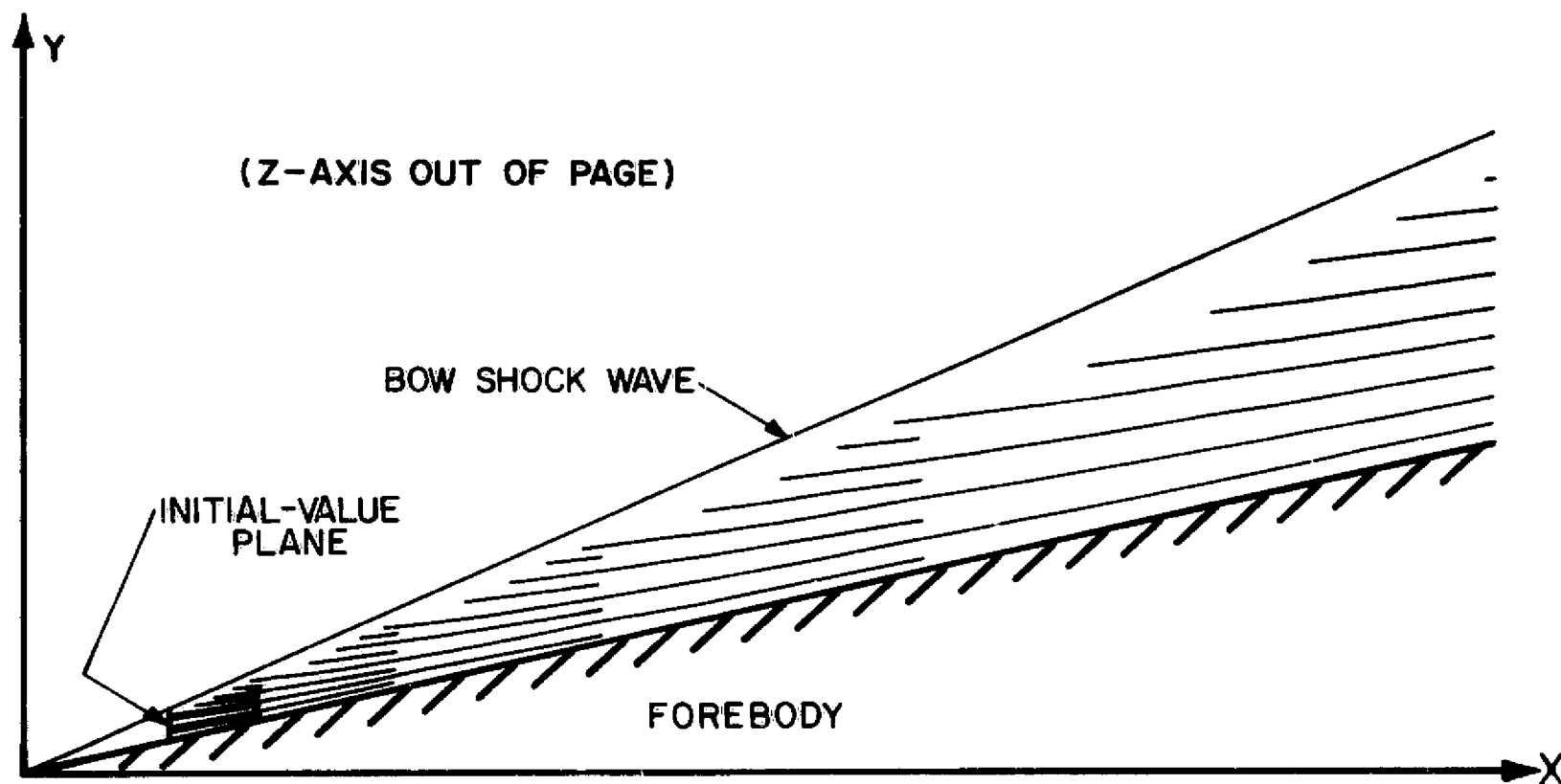


FIGURE F.4. TYPICAL FOREBODY FLOW STREAMLINE PATTERN

For the purposes of point addition, after the points on the solution plane have been computed, the mass flow rate across that plane is calculated. If this mass flow rate is significantly larger than the mass flow rate across the last solution plane where point redistribution was performed, a new ring of solution points is added between the ring of shock wave solution points ($j = n$) and the ring of outermost interior field solution points ($j = n - 1$). The coordinates of each of these inserted solution points is obtained by forming the arithmetic average of the coordinates of the shock wave solution point and the outermost interior field point corresponding to the circumferential station of the new point. The flow properties at each of the inserted solution points are obtained by interpolation using the quadratic bivariate polynomial

$$f(y,z) = a_1 + a_2y + a_3z + a_4yz + a_5y^2 + a_6z^2 \quad (F.3)$$

where $f(y,z)$ denotes a general function of the coordinates y and z . The coefficients a_i ($i=1$ to 6) in equation (F.3) are obtained by a least squares fit of nine data points in the solution plane, as described in Appendix C.

Point deletion occurs when the number of radial stations has reached a specified limit. In point deletion, the body streamline points are retained in storage, while selected interior streamline points are deleted from storage. Refinement of this technique is provided by having two limits to the number of allowable radial stations. The first limit is employed when the mass flow rate at the given solution plane is less than a specified fraction of the estimated

flow rate at the cowl lip. The second and larger limit is employed when that fraction has been exceeded.

Finally, it should be noted that the influence of molecular diffusion can be included in the forebody flow field computation.

6. INTERNAL FLOW IN WHICH SHOCK WAVES ARE NOT DISCRETELY FITTED

The program option in which the internal flow field is computed without the discrete fitting of the internal shock wave system might be employed in the cases where the internal shock waves are weak in strength, and thereby an acceptable solution could be obtained by smearing all internal discontinuities. This option requires that only two unit processes be employed: the interior point unit process and the solid boundary point unit process. The influence of molecular transport can be included in the computation of this flow field.

The initial-value plane of the internal flow computation is constituted by the last solution plane of the forebody flow field integration. Alternatively, the initial-value plane may be specified at the cowl lip axial station without employing the forebody flow field integration option. This technique is recommended if the forebody is conical up to the cowl lip axial station.

The computer program developed in the present investigation assumes that the bow shock wave falls outside of the cowl lip, or, in the limit, intersects the cowl lip exactly. The program does not have the capability to compute the internal flow field when the bow shock wave has been ingested into the annulus.

With the initial-value plane specified, a point redistribution on this plane is performed to obtain a uniform point distribution and

to obtain streamlines which lie in the stream surface formed by the cowl boundary. The redistributed points are arranged symmetrically in the computed sector. These points lie on rays which have equal angular increments from one another, with the points on each ray being spaced at equal radial increments. The radial station $j = 1$ corresponds to the centerbody streamline points, and $j = n$ corresponds to the cowl streamline points. The properties at these points are obtained by interpolation.

With the point redistribution performed, the internal flow field integration proceeds in a manner similar to the external flow field integration, except that only two unit processes are used: the interior point unit process and the solid boundary point unit process. No point addition or deletion is performed. The internal flow field integration is terminated either when a specified axial station is reached or when the flow becomes subsonic.

7. INTERNAL FLOW IN WHICH SHOCK WAVES ARE DISCRETELY FITTED

A point redistribution is first performed on the initial-value plane at the axial station of the cowl lip as described in the previous section. After the upstream flow properties have been determined at each of the cowl lip solution points in the computed sector, the downstream flow properties are obtained at each of these points by application of the solid body-shock wave point unit process.

In the integration of the internal flow field in which shock waves are discretely fitted, the axial step is obtained by the application of the CFL stability criterion, except in the vicinity of a shock wave reflection, where special constraints are employed. After the

axial station of the solution plane has been determined, the internal shock wave is projected from the current initial-value plane to the current solution plane in the meridional plane passing through the x-axis and the previous shock wave point on the initial-value plane as illustrated in Figure F.5. The location of the shock wave solution point is obtained by applying the following equation.

$$dr_s/dx = \tan\beta_I \quad (F.4)$$

In equation (F.4), dr_s is the increment in radius between the projected shock wave point and the previous shock wave point on the initial-value plane, dx is the corresponding increment in axial distance, and β_I is the angle subtended by the shock wave and the x-axis at the initial-value plane shock wave point and in the meridional plane defined by the initial-value plane shock wave point. Equation (F.4) is applied for each shock wave point in the computed sector, thereby yielding the locus of projected shock wave points in the solution plane. Interpolated values of the shock wave radius in the solution plane are obtained by employing the following equation.

$$r_s(\theta) = a_1 + a_2\theta + a_3\theta^2 \quad (F.5)$$

In equation (F.5), $r_s(\theta)$ is the shock wave radius at the polar angle $\theta = \tan^{-1}(z/y)$, and the coefficients a_i ($i=1,2,3$) are obtained by fitting this expression to three projected shock wave points, as described in Appendix C. Equation (F.5) is applied at every circumferential station in the computed sector. Hence, the shock wave location in the solution plane is represented by a series of overlapping one-dimensional curve fits.

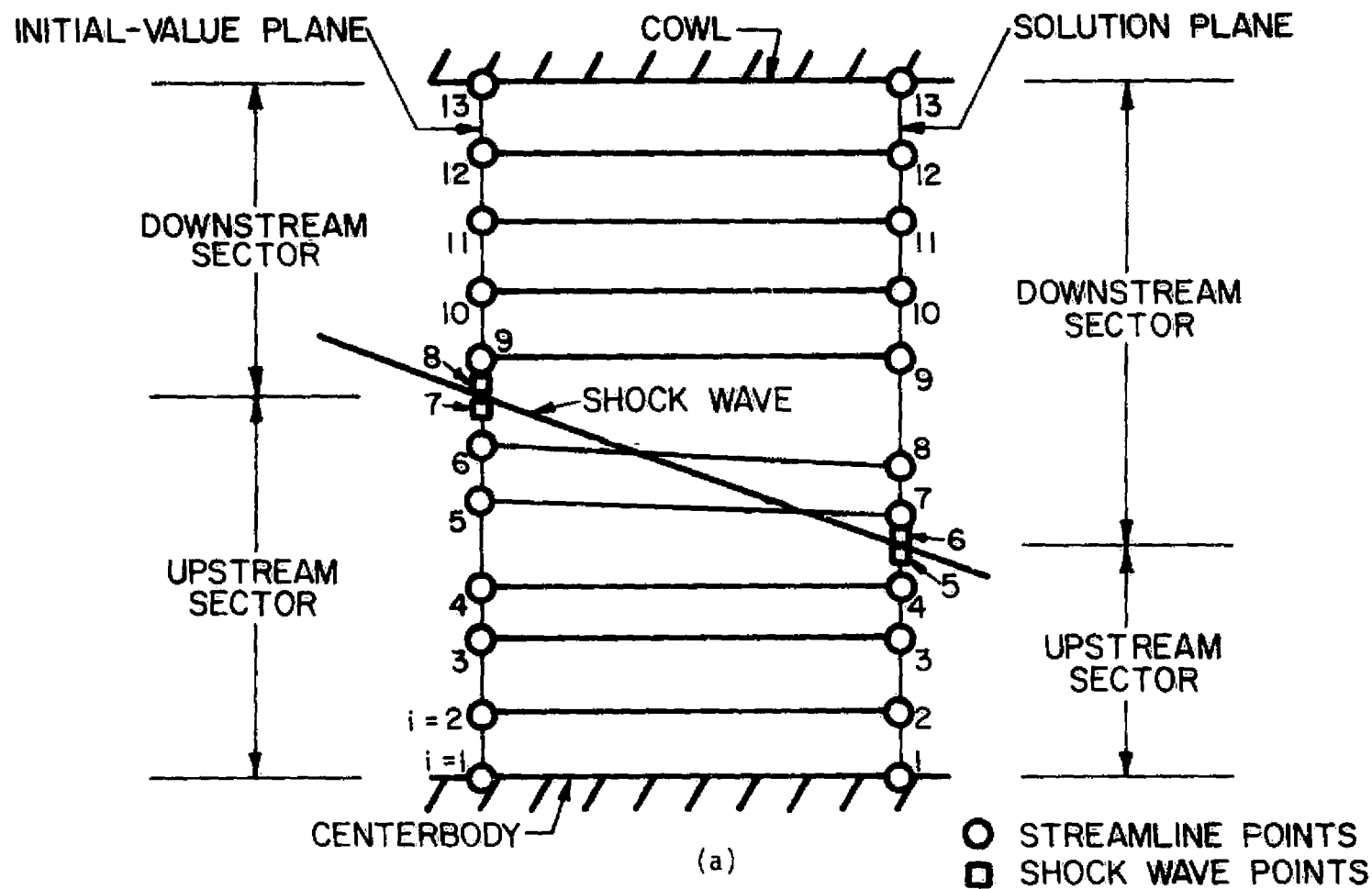
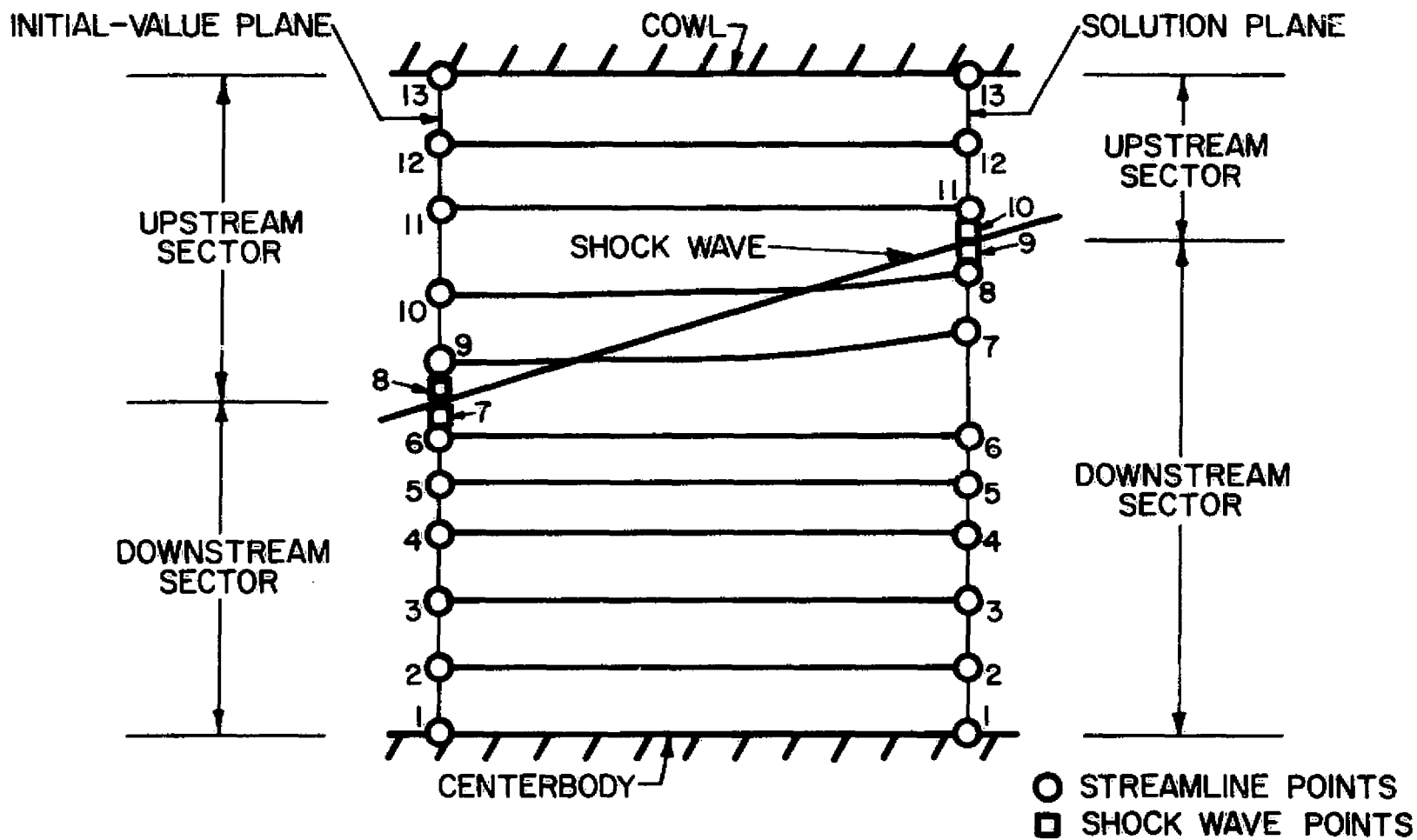


FIGURE F.5 TYPICAL STREAMLINE NETWORK FOR INTERNAL FLOW



(b)

FIGURE F.5 (CONTINUED)

After the tentative position of the shock wave in the solution plane has been determined, the streamlines that are in the flow field sector that is upstream of the shock wave in the initial-value plane are projected from the initial-value plane to the solution plane, as illustrated by streamlines 1 to 6 in Figure F.5(a) and by streamlines 9 to 13 in Figure F.5(b). This is accomplished by applying the equation of a streamline

$$dx_i = u_i dt \quad (i=1,2,3) \quad (F.6)$$

where x_i ($i=1,2,3$) denotes the three cartesian coordinates x , y , and z , respectively, u_i ($i=1,2,3$) denotes the corresponding velocity components u , v , and w , respectively, and t is the time of travel of a fluid particle along the streamline. Equation (F.6) is first applied in the x direction. Since the axial step dx is known from the application of the CFL stability criterion, the time parameter dt may be determined. Then, application of equation (F.6) for the y and z directions allows the y and z coordinates of the projected streamline point to be computed. The radius $r = (y^2 + z^2)^{1/2}$ and polar angle $\theta = \tan^{-1}(z/y)$ of each of the projected streamline points are then computed.

The radius of the projected streamline point is then compared to the radius of the shock wave, given by equation (F.5), in the meridional plane defined by the projected streamline point. If the projected streamline point is in the upstream flow field sector on the solution plane (i.e., the streamline does not intersect the shock wave), then a standard interior point or solid boundary point unit

process is applied to obtain the solution at this point. If the streamline appears to intersect the shock wave, as illustrated by streamlines 5 and 6 in Figure F.5(a) and streamlines 9 and 10 in Figure F.5(b), then the application of the unit process to determine the solution is deferred.

At this stage, the upstream and downstream shock wave solution points are determined at each circumferential station in the solution plane computed sector using the internal shock wave point unit process. This procedure defines the property field on both the upstream and downstream sides of the internal shock wave.

Next, the body streamline solution points are computed at every circumferential station in the downstream flow field sector on the solution plane. In some instances, computing the solution at these points may entail using flow property information from the downstream side of the internal shock wave if the Mach cone, with apex at the solution point, intersects the shock wave surface. Determining the solution at each of these points thereby defines the flow property field on the boundary stream surface in the downstream flow field sector.

At this stage, the solution on each of the streamlines which have not yet been computed is determined. The streamlines that are in the downstream flow field sector on the initial-value plane will remain in the downstream flow field sector on the solution plane (see Figure F.5). The solution at these points is determined by the application of the standard interior point unit process, unless a portion of the Mach cone, with apex at the solution point, intersects the internal

shock wave or the solid boundary, in which case the modified interior point unit process is applied. For streamlines which penetrate the internal shock wave [streamlines 5 and 6 in Figure F.5(a) and streamlines 9 and 10 in Figure F.5(b)], the appropriate modified interior point unit process is applied. For the streamlines whose solution was deferred due to a possible shock wave penetration, but which ultimately did not intersect the shock wave, the standard interior point scheme is applied. The solution points are ordered in the storage arrays in the order of increasing radius on a given circumferential station. So a post computation interchange of the streamline solution points with the shock wave solution points is performed for the streamlines which initially appeared to intersect the shock wave but ultimately did not.

The process just outlined is applied repetitively until the internal shock wave intersects a solid boundary. Special logic is used in the computer program for the computation of a shock wave reflection. The overall scheme used in this case is now presented.

The initial step in the computation of the shock wave-solid boundary reflection is to obtain an estimate of the axial location, at a discrete number of points, where the incident shock wave intersects the solid boundary. Except for the case of axisymmetric flow, the intersection of the incident shock wave with the solid boundary defines a three-dimensional space curve, as illustrated in Figure F.6. In axisymmetric flow, this curve lies in a plane of constant x . Points along the space curve are determined by obtaining the intersection of the shock wave and the solid boundary, where both of these

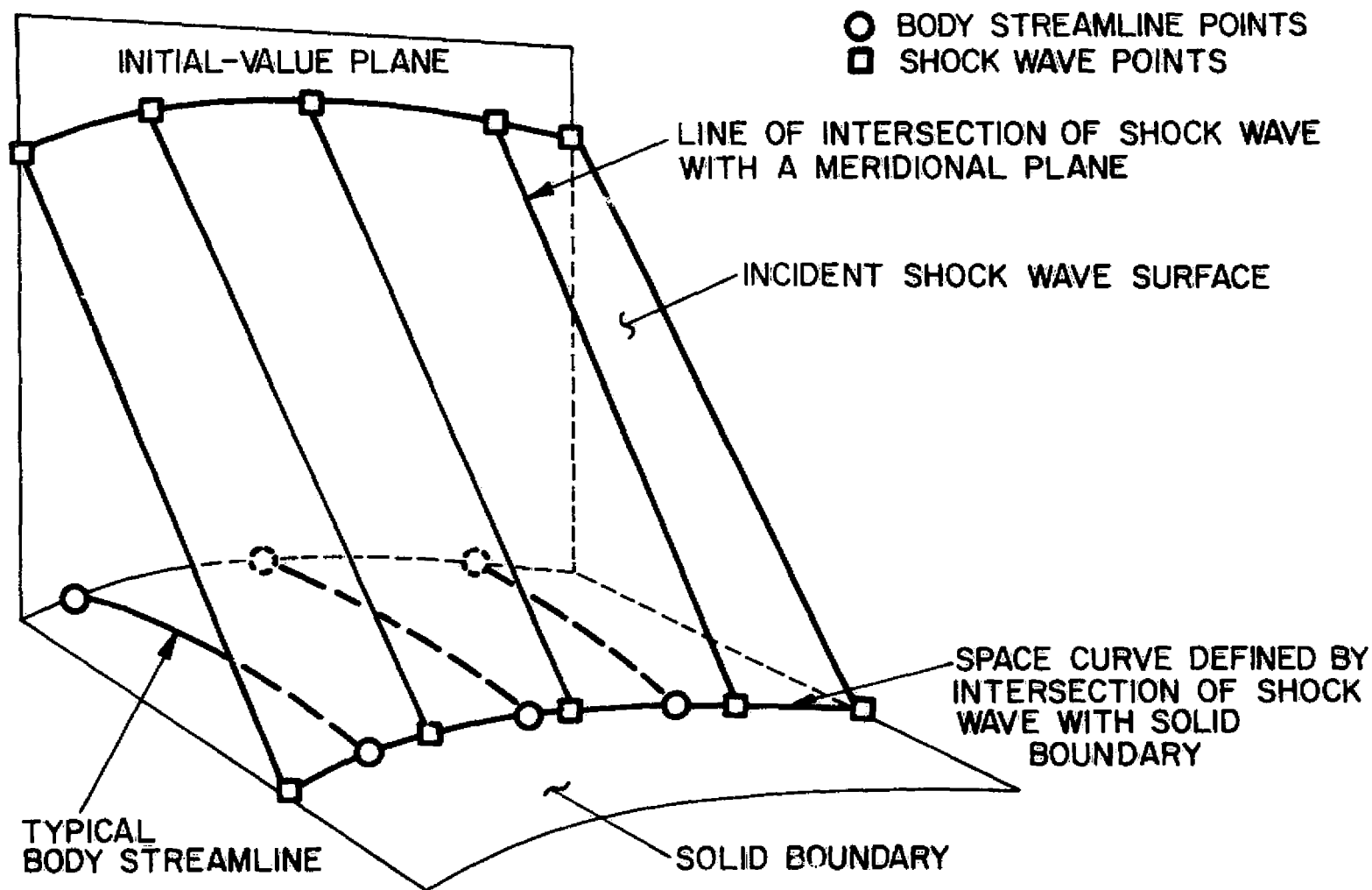


FIGURE F.6. INCIDENT SHOCK WAVE-BODY INTERSECTION

surfaces are represented as straight line segments in the meridional planes passing through the shock wave points in the initial-value plane. For a given meridional plane, the shock wave is represented by equation (F.4), where dr_s is the increment in radius between the shock wave-body intersection point and the shock wave point in the initial-value plane, dx is the corresponding increment in axial distance, and β_f is the angle subtended by the shock wave and the x-axis in the meridional plane defined by the appropriate shock wave solution point in the initial-value plane. The local body surface is approximated in the meridional plane by the equation

$$dr_b/dx = m \quad (F.7)$$

where dr_b is the change in the radius of the body between the shock wave-body intersection point and the body point in the initial-value plane, dx is the corresponding increment in axial distance, and m is the local slope of the body in the given meridional plane. Equations (F.4) and (F.7) are solved simultaneously to obtain the intersection point in the given meridional plane. The intersection point for every meridional plane defined by the shock wave points on the initial-value plane is so determined. The locus of these intersection points determines the space curve illustrated in Figure F.6.

At this stage, the points on the space curve which are nearest to and farthest away from the initial-value plane are determined. If the axial distance between the nearest point and the initial-value plane is greater than a specified fraction of the marching step allowed by the CFL stability criterion, then another solution plane

is computed, the location of this plane being just slightly upstream of the shock wave-body intersection. The entire procedure outlined above is then repeated. Alternatively, if the distance between the nearest shock wave-body intersection point and the initial-value plane is less than this fraction of the allowable marching step, then the axial position of the next solution plane is selected such that the space curve representing the incident shock wave-body intersection is entirely contained between the initial-value plane and the solution plane. At high angles of attack, this procedure may require that the axial step between the initial-value plane and the solution plane be greater than that allowed by the CFL stability criterion. This implies that the Courant number, which is the ratio of the axial step taken to the axial step allowed by the CFL stability criterion, is greater than unity. To maintain an effective Courant number less than unity, the fit point stencils used in the univariate, bivariate, and trivariate interpolation polynomials are adjusted in accord with the Courant number of the actual step taken. That is, if the Courant number is approximately two, then every other point is used in the interpolation fit point stencils instead of the immediate neighbors (which correspond to a unity Courant number), etc. This ensures that the convex hull of the finite difference network engulfs the differential domain of dependence, thereby satisfying the CFL stability criterion.

After the axial position of the solution plane has been determined and the Courant number computed, the internal shock wave point unit process is applied at every circumferential station in the computed

sector at the intersection of the incident shock wave with the solid boundary. This procedure defines the property field on both the upstream and downstream sides of the incident shock wave.

At this stage, the initial-value plane upstream sector body streamlines are extended from the initial-value plane to the space curve defined by the intersection of the incident shock wave with the solid boundary, as illustrated in Figure F.6. The solution for both the upstream and downstream shock wave properties has been obtained on the space curve by the application of the internal shock wave point unit process. Hence, both the upstream and downstream properties at the points where the body streamlines intersect the space curve may be found by interpolation. For this purpose the following quadratic univariate polynomial is employed

$$f(\theta) = a_1 + a_2\theta + a_3\theta^2 \quad (F.8)$$

where $f(\theta)$ denotes a general function of the polar angle θ . The coefficients a_i ($i=1,2,3$) in equation (F.8) are obtained by fitting this expression to three data points on the space curve as described in Appendix C. To determine the intersection point of the body streamline with the space curve, an iterative technique is used. Moreover, after each iteration, the projected streamline point is adjusted along the direction of the body normal projection in the (y,z) -plane such that the streamline point lies on the boundary surface. Equation (F.8) is applied for both the upstream and downstream shock wave properties. Hence, the incident shock wave downstream properties are known at the body streamline points.

At this stage, the solid body-shock wave point unit process is applied at each of the body streamline points in the computed sector that are on the space curve. This defines the reflected shock wave downstream properties at the body streamline points on the space curve.

Using a procedure similar to that used previously, the shock wave is then projected from the space curve to the current solution plane. This projection is performed in the meridional planes containing the body streamline points on the space curve. This procedure yields the tentative shock wave shape in the solution plane.

At this stage, the body streamline points in the solution plane that are in the downstream flow field sector in the initial-value plane are computed by use of the solid boundary point unit process (see Figure F.7). This unit process is applied at every such point in the computed sector. As a consequence, the flow property field on the stream surface formed by the solid boundary is defined.

Next, the remaining streamlines that are in the initial-value plane downstream flow field sector are projected from the initial-value plane onto the solution plane. A test is then made to determine whether or not each of these streamlines intersects the reflected shock wave (see Figure F.7). Those streamlines which do not intersect the reflected shock wave will lie in the upstream flow field sector on the solution plane (points 7 to 13 in Figure F.7). The solution at these points is determined using the standard interior point scheme, or if the Mach cone, with apex at the solution point, intersects the incident shock wave or solid boundary, the appropriate

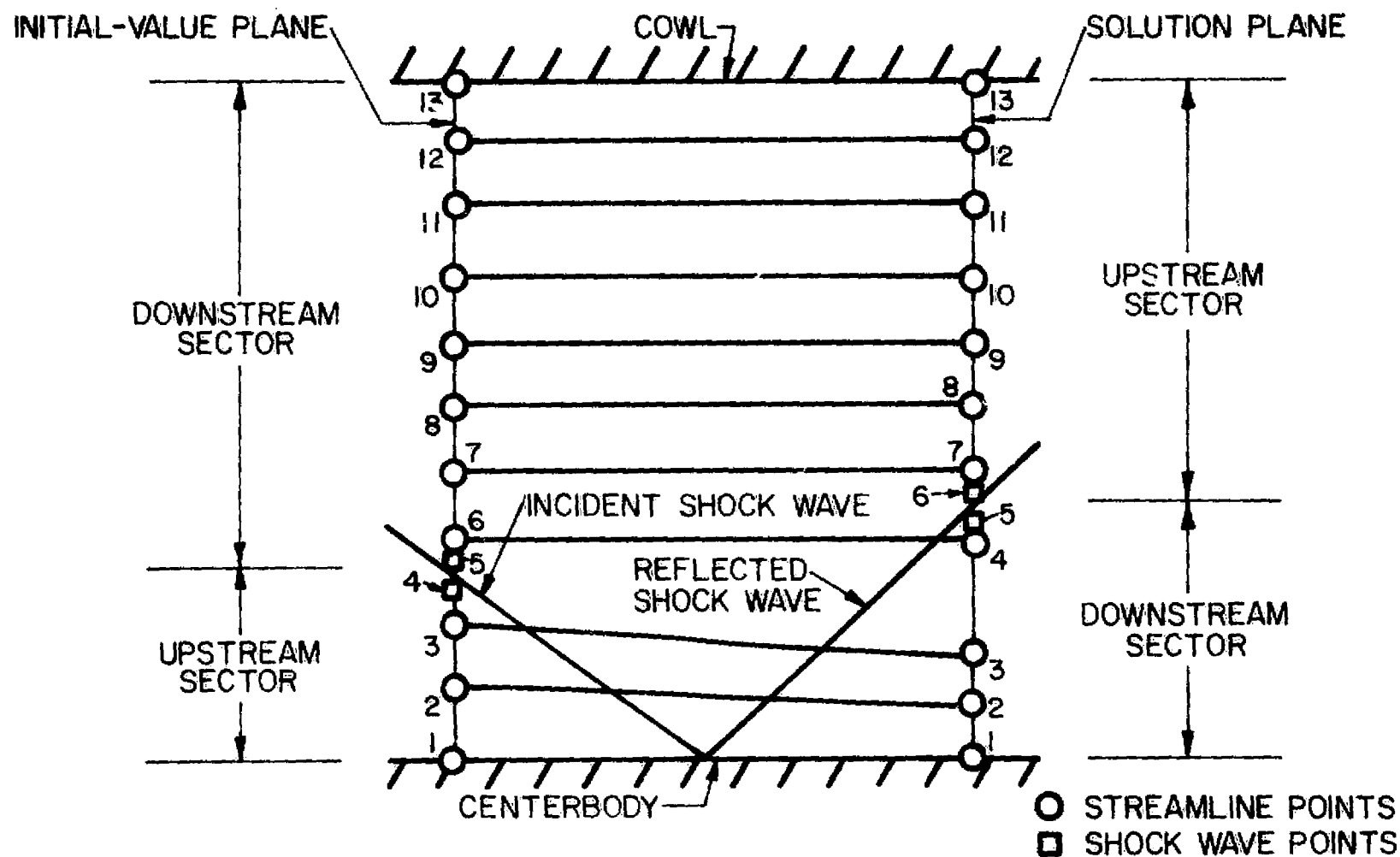


FIGURE F.7. TYPICAL STREAMLINE NETWORK AT AN INTERNAL SHOCK WAVE REFLECTION

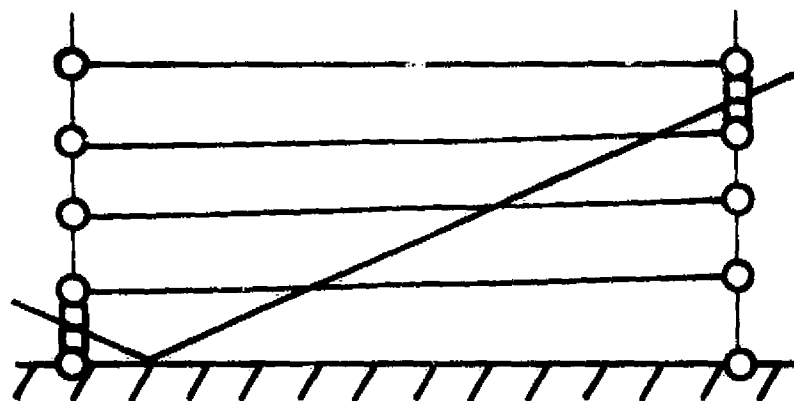
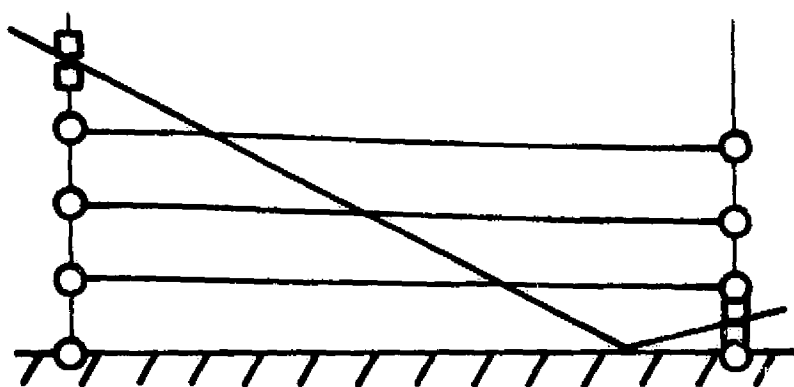
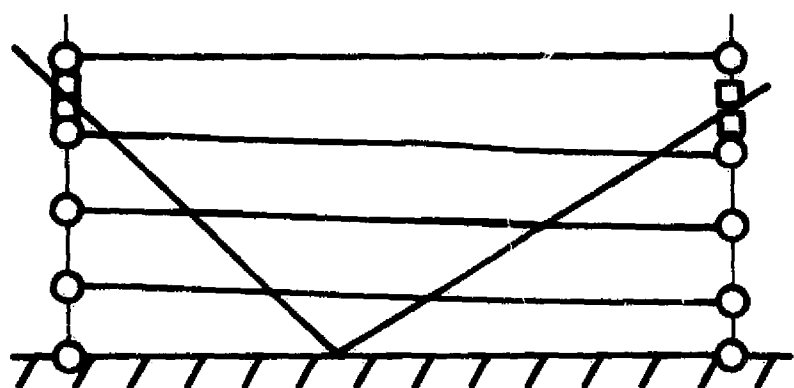
modified interior point unit process is applied. Those streamlines which appear to intersect the reflected shock wave have their computation deferred.

At this stage, the upstream and downstream shock wave points are determined at every circumferential station in the solution plane computed sector. This procedure defines the property field on both the upstream and downstream sides of the reflected internal shock wave.

Next, the solution is obtained at each body streamline point in the downstream flow field sector on the solution plane (see Figure F.7). The modified solid boundary point unit process is applied in this situation, which requires using flow property information on the downstream side of the reflected shock wave. After the application of the body point unit process at each point in the computed sector, the property field on the solid boundary is defined.

At this stage, the streamlines that are in the downstream flow field sector in the initial-value plane and that intersect the reflected shock wave are computed. These points require using the modified interior point unit process and use flow property information on both the upstream and downstream sides of the reflected internal shock wave (see Figure F.7).

Finally, the streamlines that are in the upstream flow field sector in the initial-value plane are extended to the surface of the incident shock wave and their respective intersection points with this surface are determined (see Figures F.7 and F.8). These streamlines are then extended from the downstream side of the incident shock wave to the current solution plane. If the projected streamline does not



○ STREAMLINE POINTS
 □ SHOCK WAVE POINTS

FIGURE F.8. POSSIBLE STREAMLINE PATTERNS
 AT A REFLECTION

intersect the reflected shock wave, a modified interior point unit process is applied using flow property information on the downstream side of the incident shock wave. If the projected streamline intersects the reflected shock wave, the intersection point is found with this surface. A modified interior point unit process is then applied on the downstream side of the reflected shock wave.

After all of the points have been determined on the solution plane that is immediately downstream of the shock wave-solid body reflection, control is returned to the driving algorithm until another shock wave-solid body reflection is encountered.

Figures F.6 to F.8 illustrate the intersection of the shock wave with the centerbody. Similar results hold when the shock wave intersects the cowl.

The internal flow field integration is terminated when either a specified axial station is reached or when the flow becomes subsonic.

APPENDIX G

CALCULATION OF THE TRANSPORT TERMS

1. INTRODUCTION

The numerical procedure developed in this investigation has the capability to include the influence of molecular transport on the solution by treating the viscous and thermal diffusion terms in the governing equations as forcing functions, or correction terms, in the method of characteristics scheme. At present, the computer program has the capability to include the influence of viscous and thermal diffusion in the computation of the external flow field about the forebody, and in the computation of the internal flow field in which shock waves are not discretely traced. The program option which performs discrete fitting of the internal shock wave system does not have the capability to include the influence of molecular transport in the computation, but rather assumes the flow to be inviscid and adiabatic.

2. EXPRESSIONS FOR THE TRANSPORT TERMS

The expressions for the transport forcing functions are derived in Appendix A, and are summarized below.

$$F_x = \mu_x \left[\frac{4}{3} u_x - \frac{2}{3} (v_y + w_z) \right] + \mu_y (u_y + v_x) + \mu_z (u_z + w_x) \\ + \mu \left[\frac{4}{3} u_{xx} + u_{yy} + u_{zz} + \frac{1}{3} (v_{xy} + w_{xz}) \right] \quad (G.1)$$

$$F_y = \mu_y \left[\frac{4}{3} v_y - \frac{2}{3} (u_x + w_z) \right] + \mu_x (v_x + u_y) + \mu_z (v_z + w_y) \\ + \mu \left[\frac{4}{3} v_{yy} + v_{xx} + v_{zz} + \frac{1}{3} (u_{yx} + w_{yz}) \right] \quad (G.2)$$

$$F_z = \mu_z \left[\frac{4}{3} w_z - \frac{2}{3} (u_x + v_y) \right] + \mu_x (w_x + u_z) + \mu_y (w_y + v_z) \\ + \mu \left[\frac{4}{3} w_{zz} + w_{xx} + w_{yy} + \frac{1}{3} (u_{zx} + v_{zy}) \right] \quad (G.3)$$

$$F_e = \xi \left\{ \kappa (T_{xx} + T_{yy} + T_{zz}) + \kappa_x T_x + \kappa_y T_y + \kappa_z T_z \right. \\ + \mu \left[2(u_x^2 + v_y^2 + w_z^2 + u_y v_x + u_z w_x + v_z w_y) + v_x^2 + w_x^2 \right. \\ \left. \left. + u_y^2 + w_y^2 + u_z^2 + v_z^2 - \frac{2}{3} (u_x + v_y + w_z)^2 \right] \right\} \quad (G.4)$$

where

$$\xi = \frac{1}{\rho T} \left(\frac{\partial P}{\partial s} \right)_\rho \quad (G.5)$$

In equations (G.1) to (G.5), u , v , and w denote the velocity components in the x , y , and z coordinate directions, respectively, P is the pressure, ρ denotes the density, T is the absolute temperature, s denotes the entropy per unit mass, μ represents the dynamic viscosity, and κ is the thermal conductivity. The subscripts x , y , and z on the right-hand sides of equations (G.1) to (G.4) denote partial differentiation in the corresponding coordinate direction, whereas F_x , F_y , and F_z on the left-hand sides denote the transport forcing functions in the x , y , and z component momentum equations, respectively. F_e is the transport forcing function in the energy equation.

3. COMPUTATION OF THE TRANSPORT FORCING FUNCTIONS

During the course of the program development, a number of methods were devised in an effort to obtain a good approximation to the transport forcing functions. One such method was based on employing a quadratic trivariate interpolation polynomial whose coefficients were determined by a least squares fitting of a number of known field points on the initial-value plane and the previous solution planes. This polynomial was employed to determine the five dependent properties u , v , w , P , and ρ . The spatial derivatives of the velocity components appearing in equations (G.1) to (G.4) were then obtained by analytically differentiating the respective interpolation polynomials. Spatial gradients of pressure and density were obtained in a similar manner. Then, by differentiation of the thermal equation state, temperature derivatives were expressed in terms of the pressure and density derivatives. The molecular transport properties and their spatial gradients were obtained using the procedures presented in Appendix A.

This method of calculating the transport forcing terms was considered to have good accuracy. The computer execution time required by this method, however, was felt to be unacceptable. This prohibitive execution time was primarily due to the least squares curve fitting of the trivariate interpolation polynomials. Consequently, a more efficient method with acceptable accuracy was sought for approximating the transport terms. The method which was selected is presented below.

For the interior point and solid boundary point unit processes, the transport terms must be computed at all points in the computational network (see Figures E.1 and E.2). For the bow shock

wave point unit process, the transport terms must be computed at the solution point and at the intersection point of the bicharacteristic with the initial-value plane (see Figure E.3). For each of these unit processes, partial derivatives of the dependent properties with respect to y and z in the initial-value plane are obtained by analytically differentiating the quadratic bivariate interpolation polynomial

$$f(y,z) = a_1 + a_2y + a_3z + a_4yz + a_5y^2 + a_6z^2 \quad (G.6)$$

The coefficients a_i ($i=1$ to 6) in equation (G.6) are determined by a least squares fit of nine data points in the initial-value plane as discussed in Appendix C. Equation (G.6) is applied for the five dependent flow properties u , v , w , P , and ρ . Spatial derivatives of pressure and density are required [even though they do not appear explicitly in equations (G.1) to (G.4)] because spatial derivatives of temperature are expressed in terms of pressure and density derivatives through differentiation of the thermal equation of state as discussed in Appendix A.

In the solution plane, partial derivatives of the dependent properties with respect to y and z are equated to the corresponding derivatives in the initial-value plane. For the interior point and boundary point schemes, the derivatives at the solution point are set equal to those at the streamline base point. For the bow shock wave point scheme, the solution point derivatives are equated to those at the bicharacteristic base point. The evaluation of these derivatives in the solution plane would require that a global iteration algorithm be employed. In this algorithm, the property field on the solution

plane would first be determined by a predictor application of the appropriate unit process at each point in the computed sector. Then, by fitting equation (G.6) to solution plane field points the appropriate partial derivatives could be obtained. In a similar manner, ensuing corrector applications would be performed until overall convergence was achieved. The attendant increase in algorithm complexity and computer execution time using this global iteration procedure, however, was felt to be unwarranted since the transport terms are assumed to be of secondary importance in determining the solution.

Partial derivatives with respect to x in equations (G.1) to (G.4) are obtained from the following quadratic univariate interpolation polynomial.

$$f(x) = a_1 + a_2x + a_3x^2 \quad (G.7)$$

The coefficients a_i ($i=1,2,3$) in equation (G.7) are determined by fitting this expression to three data points. The first data point is located on the solution plane that is immediately upstream of the current initial-value plane, the second data point is on the initial-value plane, and the third data point is the solution point itself. For the interior point and boundary point unit processes, the fit points are located on the streamline which passes through the solution point. For the bow shock wave point unit process, the fit points are the shock wave solution points corresponding to the circumferential index of the solution point. Special logic in the computer program takes account of point deletion and addition in the forebody flow field computation and thereby insures that the appropriate fit points are selected. Of course, for a predictor application of either the interior point or

boundary point unit processes, property values at the solution point are equated to those at the streamline base point in the initial-value plane.

Equation (G.7) is applied for the five dependent flow properties u , v , w , P , and ρ . Analytical differentiation of equation (G.7) yields approximations to the x -partial derivatives. Differentiation of the thermal equation of state allows the spatial derivatives of temperature in the x -coordinate direction to be expressed in terms of the corresponding pressure and density derivatives. This formulation yields an x -partial derivative which is constant in a given x -plane.

Since equation (G.7) uses data on a previous solution plane, derivatives cannot be evaluated using this representation until at least one previous solution plane is available. Furthermore, the derivatives obtained using this formulation are only approximations to the x -partial derivatives since the y and z coordinates of each of the three fit points are not, in general, identical. Considering that the effects of molecular diffusion are assumed to be small, this approximation is acceptable.

The molecular transport properties and their spatial gradients are obtained using the procedures presented in Appendix A.

APPENDIX H

NOMENCLATURE

ENGLISH SYMBOLS

a	sonic speed
a_i, b_i, c_i, d_i	general curve fit coefficients
B_i	body force vector in index notation
c	velocity of divergence of Mach conoid surface
e	internal energy per unit mass
f	general interpolation polynomial function
F_x, F_y, F_z, F_e	forcing functions in the x, y, and z component momentum equations and energy equation, respectively
$\hat{i}, \hat{j}, \hat{k}$	unit vectors in the x, y, and z directions, respectively
$\hat{i}', \hat{j}', \hat{k}'$	unit vectors in the x', y', and z' directions, respectively
$\hat{l} = (l_x, l_y, l_z)$	unit vector along the space curve defined by the intersection of the shock wave with either the initial-value plane or a solid boundary
M	Mach number
$\hat{n} = (n_x, n_y, n_z)$	unit vector normal to a wave surface
n_i	above unit vector in index notation
$\hat{n}_b = (n_{bx}, n_{by}, n_{bz})$	unit vector normal to a solid boundary
n_{bi}	above unit vector in index notation

$\hat{n}_s = (n_{sx'}, n_{sy'}, n_{sz'})$	unit vector normal to the shock wave surface [expressed in the (x', y', z') -system]
$\bar{N} = (N_x, N_y, N_z)$	vector normal to either a wave surface or a stream surface
p	pressure
q	velocity magnitude
r	radial position of a point
R	gas constant
R_c	cowl lip radius
R_{min}	distance from streamline base point to nearest point on convex hull
s	either entropy per unit mass, or arc length
S	temperature base in Sutherland's formula
$\bar{S} = (S_x, S_y, S_z)$	vector in the wave surface and normal to the bicharacteristic direction
t	time or time-like parameter
$\hat{t}$	unit vector along the space curve defined by the intersection of the shock wave with a meridional plane
T	absolute temperature
u, v, w	velocity components in the x , y , and z directions, respectively
u_i	velocity in index notation
$\bar{V}$	velocity vector
x, y, z	cartesian coordinates of base coordinate system
x_i	base system coordinates in index notation

x', y', z'

cartesian coordinates of local coordinate system

GREEK SYMBOLS

α	either the angle of attack, or the angle subtended by the unit vector $\hat{l}$ and the z' -axis
α_j, β_j	unit vectors used in the parameterization of the characteristic equations
γ	specific heat ratio
δ_{ij}	Kronecker delta
η	second coefficient of viscosity
θ	either the angle used in the parameterization of the characteristic equations, or the angle subtended by a meridian and the (x, y) -plane
κ	thermal conductivity
λ	term in the wave surface compatibility relation
μ	dynamic viscosity
ξ	thermodynamic parameter
ρ	density
σ	term in the noncharacteristic relation
ϕ	angle subtended by the unit vector $\hat{t}$ and the x' -axis
Φ	either the viscous dissipation function, or a term in the wave surface compatibility relation

SUBSCRIPTS

i, j, k	rectangular cartesian coordinate indices ranging from 1 to 3
x, y, z	denotes either partial differentiation with respect to x , y , and z , or the x , y , and z components of a vector
∞	free-stream conditions

OPERATORS

$D()/Dt$	material derivative
$(\vec{\quad})$	vector
$(\hat{\quad})$	unit vector

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